Analysis and Design of Hermite subdivision schemes

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Abstract

Starting from an initial sequence of Hermite elements, a Hermite subdivision scheme recursively generates finer sequences of Hermite elements which are associated with the dyadic points. With the help of the interpolating splines that can be associated with the Hermite elements, we analyze the smoothness of the limit curves generated by Hermite subdivision schemes of arbitrary order, including non–interpolatory ones. After presenting these theoretical results, we describe two new families of Hermite subdivision schemes.

Keywords. Subdivision scheme, Hermite interpolation, spline curves, curve design.

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1 Introduction

Starting from an initial sequence of Hermite elements (i.e. vectors containing function values and associated derivatives), a Hermite subdivision scheme (HSS for short) recursively generates finer sequences of Hermite elements which are associated with the dyadic points, see Dyn and Levin 1995 and 1999, Kuijt 1998, Merrien 1992. The dimension of the Hermite elements (order of derivatives+1) will be called the order of the scheme. By simultaneously designing points and associated derivatives, Hermite subdivision schemes are a useful tool for curve design. For instance, using an interpolatory scheme, curves with cusps can easily be generated, simply by choosing zero derivatives at some of the initial points.

Hermite subdivision schemes generate spaces of functions which are spanned by vectors of refinable basis functions. Such spaces, generalizing the standard wavelet constructions, have recently been discussed in approximation theory, cf. Goodman 2000, Plonka 1995, Strela and Strang 1995.

Hermite subdivision has been introduced by Merrien (1992). He studied a family of interpolatory 2–point–schemes of order 2 (i.e., dealing with function values and associated first derivatives), generating C^1 limit functions. Merrien's family of HSS generalizes the piecewise linear interpolant of the function values, and the piecewise cubic interpolant of the second order Hermite elements.

Hermite subdivision schemes can be seen as a special case of vector subdivision schemes (cf., e.g., Han and Jia 1998, Micchelli and Sauer 1997 and 1999). Unlike standard vector schemes, however, Hermite schemes have non–stationary rules, depending on the refinement level k. In order to analyze interpolatory Hermite subdivision schemes, Dyn and Levin (1995, 1999) introduced an associated point subdivision scheme, generating the divided differences of the Hermite elements. If the associated scheme is stationary, then the original HSS is said to be stationary, too. The construction of the associated point scheme requires the computation of certain matrix–valued Laurent polynomials.

Using these results, Kuijt (1998) constructed several interpolatory HSS of order 2, which were shown to generate C^2 functions. Kuijt derived the refinement rules by considering the polynomials interpolating neighboring Hermite elements, and sampling Hermite data from them.

In this paper we propose a different approach to Hermite subdivision schemes of arbitrary order, including non–interpolatory ones. We consider the interpolating splines associated with the Hermite elements. Consequently, a HSS can be associated with another subdivision scheme, generating a sequence of spline functions. An HSS will be said to be stationary, if the associated spline subdivision scheme has stationary rules. Our approach leads to a simple tools for analyzing the properties of the limit curves, and it can be used in order to design new schemes.

We provide the tools which are needed for analyzing the smoothness and differentiability of the limit curve, by generalizing the results of Dyn et al. (1991) on classical point subdivision schemes to the spline case. For the sake of brevity, the proofs will be omitted, as they are mostly similar to their counterparts in the point case. Further details can be found in the PhD thesis of the second author (Schwanecke 2000).

After presenting these theoretical results, we describe two new families of Hermite subdivision schemes. Firstly, we derive a family of interpolatory C^2 Hermite subdivision schemes which generalizes the classical 4-point scheme of Dyn et al. 1987. Secondly, using a least–squares–fitting–based approach with suitable Sobolev–type norms, we derive a new family of non–interpolatory Hermite subdivision schemes.

2 Hermite Subdivision vs. Spline Subdivision

We show that a Hermite subdivision scheme can be identified with a subdivision scheme generating a sequence of spline functions.

2.1 Hermite Subdivision Schemes (HSS)

Consider the sequence $\{\mathbf{h}_i^{(0)}\}_{i\in\mathbb{Z}}$ of m–th order Hermite elements

$$\mathbf{h}_i^{(0)} = (h_{i,0}^{(0)}, h_{i,1}^{(0)}, \ldots, h_{i,m-1}^{(0)})^\top, \tag{1}$$

i.e. vectors containing function values $h_{i,0}^{(0)}$ and associated derivatives $h_{i,1}^{(0)}, \ldots, h_{i,m-1}^{(0)}$ up to order $m-1$ of an (unknown) function $f^{(0)}(t)$ at the integers,

$$\frac{d^j}{dx^j} f^{(0)}(t)\bigg|_{t=i} = h_{i,j}^{(0)}. \tag{2}$$

Starting from this initial sequence, a *Hermite subdivision scheme (HSS) of order m* recursively generates finer sequences $\{\mathbf{h}_i^{(k)}\}_{i\in\mathbb{Z}}$ of Hermite elements. The k–th generation of Hermite elements is associated with the dyadic points $\{t_i^{(k)} = i\,2^{-k}\}_{i\in\mathbb{Z}}$.

The refinement is based on the two rules,

$$\mathbf{h}_{2i}^{(k+1)} = \sum_{j=0}^{l} A_j^{(k)} \mathbf{h}_{i+j}^{(k)}, \quad \mathbf{h}_{2i+1}^{(k+1)} = \sum_{j=0}^{l} B_j^{(k)} \mathbf{h}_{i+j}^{(k)}, \quad k = 0, 1, 2, \ldots, \tag{3}$$

with the matrix masks

$$\mathbf{A}^{(k)} = \{A_0^{(k)}, \ldots, A_l^{(k)}\} \quad \text{and} \quad \mathbf{B}^{(k)} = \{B_0^{(k)}, \ldots, B_l^{(k)}\} \tag{4}$$

which consist of real $m \times m$ matrices $A_j^{(k)}, B_j^{(k)}$. Note that the masks may depend on the subdivision level k.

The sequence of Hermite elements is said to converge to a C^r limit function $f(t)$, if

$$\frac{d^j}{dx^j} f(t)\bigg|_{t=t_i^{(k)}} = \lim_{q\to\infty} h_{2^q\,i,j}^{(k+q)} \quad j = 0, \ldots, \min\{r, m-1\} \tag{5}$$

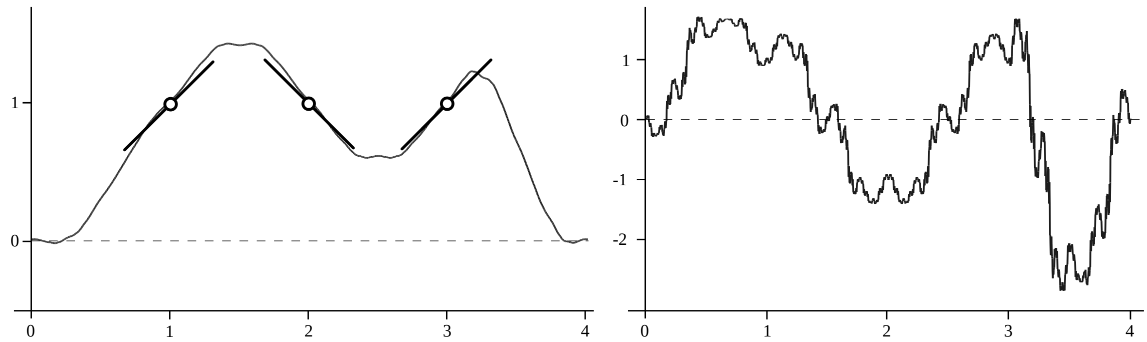

Figure 1: A curve (left) generated by Merrien's interpolating HSS for $(\alpha, \beta) = (\frac{1}{5}, \frac{9}{5})$, along with its first derivative (right).

holds for all $i, k \in \mathbb{Z}$, $k \geq 0$.

Example. An interesting family of HSS of order 1 with $l = 1$, depending on two parameters α, β, has been introduced by Merrien 1992. It is described by the matrix masks

$$A_0^{(k)} = I_{2\times 2},\ A_1^{(k)} = 0_{2\times 2},\ B_0^{(k)} = \begin{pmatrix} \frac{1}{2} & \frac{\alpha}{2^k} \\ -\beta 2^k & \frac{1-\beta}{2} \end{pmatrix},\ B_1^{(k)} = \begin{pmatrix} \frac{1}{2} & -\frac{\alpha}{2^k} \\ \beta 2^k & \frac{1-\beta}{2} \end{pmatrix}. \tag{6}$$

For $(\alpha, \beta) = (0, 1)$, the scheme produces simply the piecewise linear function interpolating the data $(i, h_{i,0}^{(0)})$.

For $(\alpha, \beta) = (\frac{1}{8}, \frac{3}{2})$, the limit function is C^1. The sequence of Hermite elements converges to the unique cubic Hermite spline matching the initial first order Hermite elements $(i, \mathbf{h}_i^{(0)})$.

For general values of the parameters α, β, the scheme produces sequences of Hermite elements which interpolate the initial data, due to choice of the matrices $A_i^{(k)}$. In particular, Merrien has proposed to choose $\beta = 4\alpha + 1$, see Merrien 1992 for details. An example is shown in Figure 1.

In this paper, we will use Merrien's family of HSS in order to illustrate the principal ideas. In particular, a more detailed analysis of the differentiability of the limit curve will be given. $\Diamond$

2.2 Associated Spline Function

The sequence $\{\mathbf{h}_i^{(k)}\}_{i\in\mathbb{Z}}$ of m–th order Hermite elements is interpolated with a C^{m-1} spline function $X^{(k)}(t)$ of order $2m$,

$$X^{(k)}(t) = \sum_{i\in\mathbb{Z}} p_i^{(k)}\ N_{i,2m}(t), \quad t \in \mathbb{R}. \tag{7}$$

The B-splines $N_{i,2m}(t)$ are defined over the knot vector

$$\mathcal{T}^{(k)} = (\ldots, \{t_{i-1}^{(k)}\}^{\langle m\rangle}, \{t_i^{(k)}\}^{\langle m\rangle}, \{t_{i+1}^{(k)}\}^{\langle m\rangle}, \ldots) \tag{8}$$

where all knots have multiplicity m, as indicated by $\{.\}^{\langle m\rangle}$. The coefficients (control points) $p_i^{(k)}$ are associated with the corresponding Greville abscissas

$$\xi_{im+j}^{(k)} = t_i^{(k)} + \frac{1-m+2j}{2^k(2m-1)}, \quad j=0,\dots,m-1,\ i\in\mathbb{Z},\ k=0,1,2,\dots. \tag{9}$$

These abscissas divide the real axis in segments according to the ratios

$$\dots:2:2:1:\underbrace{2:2:\dots:2}_{m-1 \text{ times}}:1:2:2:\dots \tag{10}$$

See e.g. Schumaker 1981 for further information on B-spline and Greville abscissas.

The spline function (7) is to match the associated Hermite elements, i.e.

$$\left.\frac{d^j}{dx^j}X^{(k)}(t)\right|_{t=t_i^{(k)}} = h_{i,j}^{(k)}. \tag{11}$$

The Hermite data $\mathbf{h}_i^{(k)}$ and the control points $p_i^{(k)}$ are related by a linear transformation.

Lemma 1 *The Hermite elements $\mathbf{h}_i^{(k)}$ of order m and the coefficients $p_i^{(k)}$ of the associated spline function $X^{(k)}(t)$ are related by*

$$\begin{pmatrix} p_{im}^{(k)} \\ \vdots \\ p_{im+(m-1)}^{(k)} \end{pmatrix} = H^{(k)}\mathbf{h}_i^{(k)} \quad \textit{resp.} \quad \mathbf{h}_i^{(k)} = \left(H^{(k)}\right)^{-1} \begin{pmatrix} p_{im}^{(k)} \\ \vdots \\ p_{im+(m-1)}^{(k)} \end{pmatrix}, \tag{12}$$

where $H^{(k)} = VU^{(k)}$ resp. $\left(H^{(k)}\right)^{-1} = \left(U^{(k)}\right)^{-1}V^{-1}$ with the lower triangular matrices

$$\begin{aligned} U^{(k)} &= \left(\frac{1}{2^{(j-1)k}}\frac{(2m-j)!}{(2m-1)!}\binom{i-1}{j-1}\right)_{j\le i=1,\dots,m} \\ \left(U^{(k)}\right)^{-1} &= \left((-1)^{i-j}2^{(i-1)k}\frac{(2m-1)!}{(2m-i)!}\binom{i-1}{j-1}\right)_{j\le i=1,\dots,m}, \end{aligned}$$

and the upper triangular matrices

$$V = \left((-1)^{j-i}\binom{m-i}{j-i}2^{m-j}\right)_{i\le j=1,\dots,m}, \quad V^{-1} = \left(\binom{m-i}{j-i}2^{i-m}\right)_{i\le j=1,\dots,m}.$$

Example. For Hermite elements of order $m=1$ we obtain the piecewise linear (spline) function interpolating the data $(t_i^{(k)}, h_{i,0}^{(k)})$. In this case, Hermite data and control points are identical.

For Hermite elements of order $m=2$, the Greville abscissas of the associated cubic spline are $\xi_{2i}^{(k)} = t_i^{(k)} - \frac{1}{3\cdot 2^k}$ and $\xi_{2i+1}^{(k)} = t_i^{(k)} + \frac{1}{3\cdot 2^k}$. The Hermite elements and the control points are related by the transformation matrices

$$H^{(k)} = \begin{pmatrix} 1 & -\frac{1}{3\cdot 2^k} \\ 1 & \frac{1}{3\cdot 2^k} \end{pmatrix} \quad \text{and} \quad {H^{(k)}}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -3\cdot 2^{k-1} & 3\cdot 2^{k-1} \end{pmatrix}, \quad k=0,1,2,\dots.$$

Both cases are shown in Figure 2. ◊

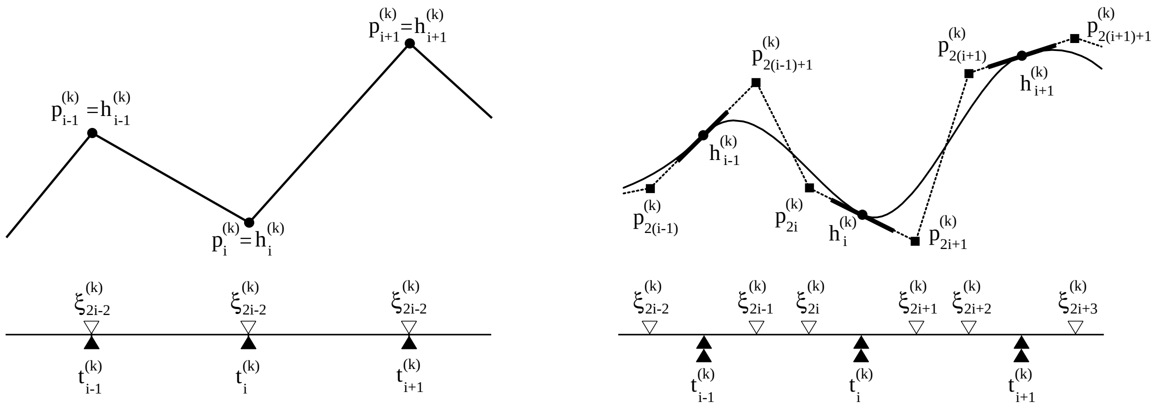

Figure 2: Hermite elements and associated spline function in B-spline form for $m = 1$ (left) and $m = 2$ (right). In the latter case, the knots of the spline function have multiplicity 2 (black triangles), and the control points (black squares) are associated with the non–uniform Greville abscissas (hollow triangles).

2.3 Associated Spline Subdivision Scheme (SSS)

The recurrence relations (3) of the Hermite elements imply a recurrence relation for the control points of the associated spline functions. Combining Lemma 1 with (3) we obtain the following result.

Proposition 2 *Consider a Hermite subdivision scheme (3) of order m, generating sequences of Hermite elements $\{\mathbf{h}_i^{(k)}\}_{i\in\mathbb{Z}}$. Assume that the matrices*

$$\hat{A}_j := H^{(k+1)} A_j^{(k)} {H^{(k)}}^{-1} \text{ and } \hat{B}_j := H^{(k+1)} B_j^{(k)} {H^{(k)}}^{-1},\ j = 0,\dots,l, \tag{13}$$

do not depend on the subdivision level k. The control points of the associated spline functions (7) satisfy the $2m$ recurrence relations

$$p_{2mi+h}^{(k+1)} = \sum_{r=0}^{m(l+1)-1} a_r^h p_{im+r}^{(k)}, \quad h = 0,\dots,2m-1, \quad k = 0,1,2,\dots\ , \tag{14}$$

where the coefficients are obtained from (13) via

$$\hat{A}_j = \begin{pmatrix} a_{mj}^0 & \cdots & a_{m(j+1)-1}^0 \\ \vdots & & \vdots \\ a_{mj}^{m-1} & \cdots & a_{m(j+1)-1}^{m-1} \end{pmatrix}, \text{ and } \hat{B}_j = \begin{pmatrix} a_{mj}^m & \cdots & a_{m(j+1)-1}^m \\ \vdots & & \vdots \\ a_{mj}^{2m-1} & \cdots & a_{m(j+1)-1}^{2m-1} \end{pmatrix}. \tag{15}$$

For any given set of $m \times m$ coefficient matrices $\hat{A}_j$, $\hat{B}_j$, $j = 0,\dots,l$, the recurrence relation 14 can be used to generate a sequence of spline functions $X^{(k)}(t)$, $k = 1,2,\dots$, starting from an initial function $X^{(0)}(t)$. This process will be called a (stationary) spline subdivision scheme (SSS) $\mathcal{S} = \mathcal{S}(\hat{A}_0,\dots,\hat{A}_l,\hat{B}_0,\dots,\hat{B}_l)$ of order $2m$.

As a technical assumption, we always require that $(A_0, B_0) \neq (0_{m\times m}, 0_{m\times m}) \neq (A_l, B_l)$, i.e., l is assumed to be as small as possible.

If the assumption of the Proposition is satisfied, i.e., if the matrices $\hat{A}_j$, $\hat{B}_j$ do not depend on the subdivision level k, then the HSS will be said to be stationary. This definition can be shown to be equivalent to the definition of a stationary interpolatory HSS given in Dyn and Levin 1995 and 1999, see Schwanecke 2000. As observed in the proposition, any stationary HSS is equivalent with a stationary SSS, and the subdivision masks are related by (13).

Example. In the case of Merrien's Hermite subdivision scheme (where $m = 2$ and $l = 1$), we obtain from (13)

$$\hat{A}_0 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \; \hat{A}_1 = 0_{2\times 2}, \; \hat{B}_0 = \begin{pmatrix} \frac{9-36\alpha-\beta}{24} & \frac{3+36\alpha+5\beta}{24} \\ \frac{3-36\alpha+\beta}{24} & \frac{9+36\alpha-5\beta}{24} \end{pmatrix}, \; \hat{B}_1 = \begin{pmatrix} \frac{9+36\alpha-5\beta}{24} & \frac{3-36\alpha+\beta}{24} \\ \frac{3+36\alpha+5\beta}{24} & \frac{9-36\alpha-\beta}{24} \end{pmatrix}. \tag{16}$$

Consequently, the associated SSS of order $2m = 4$ is given by the subdivision rules

$$\begin{aligned} p_{4i}^{(k+1)} &= \tfrac{3}{4} p_{2i}^{(k)} + \tfrac{1}{4} p_{2i+1}^{(k)}, \\ p_{4i+1}^{(k+1)} &= \tfrac{1}{4} p_{2i}^{(k)} + \tfrac{3}{4} p_{2i+1}^{(k)}, \\ p_{4i+2}^{(k+1)} &= \tfrac{9-36\alpha-\beta}{24} p_{2i}^{(k)} + \tfrac{3+36\alpha+5\beta}{24} p_{2i+1}^{(k)} + \tfrac{9+36\alpha-5\beta}{24} p_{2i+2}^{(k)} + \tfrac{3-36\alpha+\beta}{24} p_{2i+3}^{(k)}, \\ p_{4i+3}^{(k+1)} &= \tfrac{3-36\alpha+\beta}{24} p_{2i}^{(k)} + \tfrac{9+36\alpha-5\beta}{24} p_{2i+1}^{(k)} + \tfrac{3+36\alpha+5\beta}{24} p_{2i+2}^{(k)} + \tfrac{9-36\alpha-\beta}{24} p_{2i+3}^{(k)}. \end{aligned} \tag{17}$$

This scheme generates a sequence of cubic spline functions $X^{(k)}(t)$ with double knots at the dyadic points $t_i^{(k)}$. Note that the coefficients of each rule sum up to one. This will turn out to be a necessary conditions for convergence of an SSS. ◊

3 Analysis

A powerful method for analyzing (point–) subdivision schemes has been described in Dyn et al. 1991. We will adapt these techniques to the more general class of SSS.

Consider an SSS of order $2m$ on the finite domain $I = [0, n] \subset \mathbb{R}$ with some fixed $n \in \mathbb{Z}_+$. It is well–defined for all $k \geq 0$ if the associated Hermite elements of level k are defined for all dyadic points $\{t_i^{(k)} | i = 0, \ldots, Z^{(k)}\}$, where

$$Z^{(k)} = 2^k n + n_1, \quad \text{with} \quad n_1 = \begin{cases} 2l-1 & \text{if } \hat{A}_l \neq 0 \\ 2l-2 & \text{otherwise.} \end{cases} \tag{18}$$

The associated spline function is then of the form

$$X^{(k)}(t) = \sum_{i=0}^{\hat{Z}^{(k)}} p_i^{(k)} N_{i,2m,\mathcal{T}^{(k)}}(t), \quad \text{where} \quad \hat{Z}^{(k)} = m(2^k n + n_1) + m - 1, \tag{19}$$

with the knots

$$\mathcal{T}^{(k)} = (\{t_{-1}^{(k)}\}^{\langle m\rangle}, \{t_0^{(k)}\}^{\langle m\rangle}, \ldots, \{t_{2^k n+n_1+1}^{(k)}\}^{\langle m\rangle}).$$

3.1 Generator Matrix

We consider all control points of the spline function $X^{(k)}(t)$ which determine the future behaviour of the sequence of spline functions in the interval $[t_i^{(k)}, t_{i+1}^{(k)}]$. These control points will be gathered in the control point vector

$$\mathbf{p}_i^{(k)} = (p_{mi}^{(k)}, \ldots, p_{m(i+n_1+2)-1}^{(k)}). \tag{20}$$

The evolution of the control point vectors can be described with the help of the generator matrix G of the SSS. If $\hat{A}_l \neq 0$, the matrix G is defined as

$$G = \begin{pmatrix} \hat{A}_0 & \cdots & \cdots & \hat{A}_l & 0_{m\times m} & \cdots & \cdots & 0_{m\times m} \\ \hat{B}_0 & \cdots & \cdots & \hat{B}_l & 0_{m\times m} & \cdots & \cdots & 0_{m\times m} \\ 0_{m\times m} & \hat{A}_0 & \cdots & \cdots & \hat{A}_l & 0_{m\times m} & \cdots & 0_{m\times m} \\ 0_{m\times m} & \hat{B}_0 & \cdots & \cdots & \hat{B}_l & 0_{m\times m} & \cdots & 0_{m\times m} \\ \vdots & \ddots & \ddots & & & \ddots & \ddots & \vdots \\ 0_{m\times m} & \cdots & 0_{m\times m} & \hat{A}_0 & \cdots & \cdots & \hat{A}_l & 0_{m\times m} \\ 0_{m\times m} & \cdots & 0_{m\times m} & \hat{B}_0 & \cdots & \cdots & \hat{B}_l & 0_{m\times m} \end{pmatrix}. \tag{21}$$

Otherwise, the matrix is the same as before, but with the last m rows and columns deleted. G is a $M \times M$ matrix with $M = m(n_1 + 3)$.

The control point vectors satisfy the linear recurrence relations

$$\mathbf{p}_{2i}^{(k+1)} = G_0\, \mathbf{p}_i^{(k)} \quad \text{and} \quad \mathbf{p}_{2i+1}^{(k+1)} = G_1\, \mathbf{p}_i^{(k)} \tag{22}$$

where $G_0 = G\binom{1 \cdots M-m}{1 \cdots M-m}$ and $G_1 = G\binom{1+m \cdots M}{1 \cdots M-m}$. Here, we denote by

$$G\begin{pmatrix} a & \cdots & b \\ c & \cdots & d \end{pmatrix} = (g_{i,j})_{i=a,\ldots,b;j=c,\ldots,d} \tag{23}$$

the submatrix of the generator matrix $G = (g_{i,j})_{i,j=1,\ldots,M}$, consisting of the rows $a, \ldots, b$ and the columns $c, \ldots, d$.

Suppose that a parameter value $t_i^{(k)}$ has the dyadic representation

$$t_i^{(k)} = i_0 + \sum_{j=1}^{k} i_j 2^{-j} \in [0, n - 2^{-k}] \tag{24}$$

with $i_0 = \lfloor t_i^{(k)} \rfloor$ and $\{i_1, \ldots i_k\} \subseteq \{0, 1\}$. Resulting from (22), the associated control point vector can be expressed in terms of the initial data,

$$\mathbf{p}_{i,k} = G_{i_k} \cdots G_{i_1} \mathbf{p}_{i_0,0}. \tag{25}$$

Technical Remark. Without loss of generality we may assume that the coefficients of the SSS satisfy $\hat{B}_0 \neq 0_{m\times m}$. If this assumption is not satisfied by the original scheme, one should consider the SSS which is obtained by applying a shift of m to the control points. The same technique has been used in Dyn et al. 1991, Proposition 2.1.

Example. The SSS which is associated with Merrien's scheme ($m = 2$, $l = 1$) can equivalently be described by the 6×6 generator matrix

$$G = \begin{pmatrix} \hat{A}_0 & \hat{A}_1 & 0_{2\times 2} \\ \hat{B}_0 & \hat{B}_1 & 0_{2\times 2} \\ 0_{2\times 2} & \hat{A}_0 & \hat{A}_1 \end{pmatrix}, \tag{26}$$

cf. (16). Note that $\hat{A}_1 = 0_{2\times 2}$, hence $n_1 = 2l - 2 = 0$. Consequently, G is a 6×6 matrix. The control point vectors $\mathbf{p}_i^{(k)} = (p_{2i}^{(k)}, \ldots, p_{2i+3}^{(k)})$ have length 4; they satisfy the recurrence (22) with the two 4×4 submatrices

$$G_0 = \begin{pmatrix} \hat{A}_0 & \hat{A}_1 \\ \hat{B}_0 & \hat{B}_1 \end{pmatrix} \quad \text{and} \quad G_1 = \begin{pmatrix} \hat{B}_0 & \hat{B}_1 \\ 0_{2\times 2} & \hat{A}_0 \end{pmatrix} \quad \Diamond \tag{27}$$

3.2 Convergence

The sequence of spline functions $X^{(k)}(t)$ generated by an SSS is said to converge uniformly to a C^s limit function f on $[0, n]$, if

$$\begin{aligned} &\forall \epsilon > 0 \quad \exists K = K(\epsilon) : \\ &\qquad k \geq K \quad \Rightarrow \quad \forall i = 0, \ldots, m(2^k n + 1) - 1 : \quad |f(\xi_i^{(k)}) - p_i^{(k)}| \leq \epsilon. \end{aligned} \tag{28}$$

According to this definition, the sequence of control polygons (which may be considered as piecewise linear functions with respect to the Greville abscissas (9), but it suffices to consider the control points) converges uniformly to the C^s limit function f on $[0, n]$. Due to the uniform convergence of the control polygon to the B-spline curve, this implies the uniform convergence of the associated sequence of spline curves $X^{(k)}(t)$.

3.3 Continuity of the Limit Curve

The continuity of the limit curve can be analyzed as in Dyn et al. 1991, Section 3. In this section we give an outline of the main results (see Schwanecke 2000 for further details).

Necessary conditions. Firstly, if an SSS of order $2m$ converges uniformly to a continuous limit function $f \not\equiv 0$ on $[0, n]$, for arbitrary initial data $X^{(0)}(t)$, then

$$\sum_{j=0}^{m(l+1)-1} a_j^h = 1 \quad \text{for} \quad h = 0, \ldots, 2m - 1. \tag{29}$$

That is, the components of each row of the generator matrix (21) sum up to one. If this condition is satisfied, then the SSS is guaranteed to reproduce constant functions, i.e. $p_0^{(k)} = \ldots = p_{Z^{(k)}}^{(k)} = C$ implies $p_0^{(k+1)} = p_{Z^{(k+1)}}^{(k+1)} = C$.

Secondly, each of the new coefficients $p_i^{(k+1)}$ has to depend on 'sufficiently many' old coefficients. More precisely, the submatrices G_0 and G_1 governing the transformation of the control point vectors have to have at least one common row,

$$M - m \geq 1 + m, \quad \text{or, equivalently,} \quad m(n_1 + 1) \geq 1. \tag{30}$$

For instance, in the special case of a point subdivision scheme ($m = 1$), a scheme with $l = 0$ will never be C^0.

In the remainder of this paper we assume that both necessary conditions are satisfied.

Difference process. Now consider the differences of adjacent control points,

$$\Delta_i^{(k)} = p_{i+1}^{(k)} - p_i^{(k)}, \quad i = 0, \ldots, \hat{Z}^{(k)} - 1. \tag{31}$$

Suppose there exists an integer $L > 0$ and an $\alpha \in [0, 1)$, such that

$$\max_{i=0,\ldots,\hat{Z}^{(k+L)}-1} |\Delta_i^{(k+L)}| \leq \alpha \max_{i=0,\ldots,\hat{Z}^{(k)}-1} |\Delta_i^{(k)}|. \tag{32}$$

Then the spline subdivision scheme $\mathcal{S}(\hat{A}_0, \ldots, \hat{A}_l, \hat{B}_0, \ldots, \hat{B}_l)$ converges uniformly to a continuous function f on $[0, n]$. Thus, if the differences are contracting, then the sequence of spline functions is a Cauchy sequence, converging to a continuous limit function.

Next we analyze the subdivision scheme which generates the differences of control points. If the necessary conditions (29),(30) are satisfied, then the differences can be generated by another subdivision scheme

$$\Delta\mathcal{S} = \Delta\mathcal{S}(\hat{A}_0, \ldots, \hat{A}_l, \hat{B}_0, \ldots, \hat{B}_l). \tag{33}$$

Again we use the generator matrix and control point vectors in order to describe this scheme. The $(M-1) \times (M-1)$ matrix

$$C = E_M \, G \, (E_M)^{-1} \begin{pmatrix} 1, \ldots, M-1 \\ 1, \ldots, M-1 \end{pmatrix} \tag{34}$$

is the generator matrix of the difference process (33), where

$$E_M = (-\delta_{i,j} + \delta_{i+1,j}) \;\text{ and }\; (E_M)^{-1} = (-\sum_{h=0}^{M} \delta_{i+h,j} + \delta_{i+1,j}). \tag{35}$$

The proof is similar that of Proposition 3.1 of Dyn et al. 1991. Due to the necessary conditions (29), (30), the differences of the control points at each level can be expressed by differences of control points at the previous level. A short calculation confirms that the resulting subdivision process is governed by the above generator matrix C, cf. Schwanecke 2000, p.87.

Now consider the difference vectors of length $M-(m+1)$,

$$\mathbf{\Delta}_i^{(k)} = (\Delta_{mi}^{(k)}, \ldots, \Delta_{m(i+n_1+2)-2}^{(k)})^\top.$$

These vectors control the future behaviour of the difference process in the interval $[t_i^{(k)}, t_{i+1}^{(k)}]$. The difference scheme (33) can be seen as a transformation acting on these difference vectors,

$$\mathbf{\Delta}_{2i,k+1} = C_0 \mathbf{\Delta}_{i,k} \quad \text{and} \quad \mathbf{\Delta}_{2i+1,k+1} = C_1 \mathbf{\Delta}_{i,k},$$

where $C_0 = C\binom{1 \cdots M-(m+1)}{1 \cdots M-(m+1)}$ and $C_1 = C\binom{1+m \cdots M-1}{1 \cdots M-(m+1)}$. Similar to (25), any difference vector can be expressed in terms of the initial data $\mathbf{\Delta}_i^{(0)}$,

$$\mathbf{\Delta}_i^{(k)} = C_{i_k} \cdots C_{i_1} \mathbf{\Delta}_{i_0}^{(0)}, \tag{36}$$

with the dyadic representation (24).

Continuity criterion. The following result will be used as the main tool for analyzing the continuity of the limit function.

Theorem 3 (Continuity) *Let the spline subdivision scheme $\mathcal{S}$ satisfy the necessary conditions (29),(30). Then the following are equivalent:*

1. *The scheme $\mathcal{S}$ converges uniformly to a continuous limit function on $[0,n]$ for arbitrary initial data.*
2. *The difference process $\Delta\mathcal{S}$ converges uniformly to zero on $[0,n]$ for arbitrary initial data.*
3. *There exists an integer $L > 0$ and an $\alpha \in [0,1)$, such that*

$$\|C_{i_1} \cdots C_{i_L}\|_\infty \le \alpha \quad \forall \; i_j \in \{0,1\}, \; j = 1, \ldots, L. \tag{37}$$

Thus, in order to prove the continuity of the limit function, the difference process has to be shown to be contractive, by collecting several steps of the scheme.

Example. The difference scheme of the SSS associated with Merrien's HSS has the generator matrix

$$C = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{-3+36\alpha+\beta}{24} & \frac{3-\beta}{6} & \frac{3-36\alpha+\beta}{24} & 0 & 0 \\ \frac{3-\beta}{12} & \frac{\beta}{3} & \frac{3-\beta}{12} & 0 & 0 \\ \frac{3-36\alpha+\beta}{24} & \frac{3-\beta}{6} & \frac{-3+36\alpha+\beta}{24} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \tag{38}$$

see (26) and (34). The transformation of the difference vectors $\mathbf{\Delta}_i^{(k)}$ is governed by the two submatrices $C_0 = \binom{1\cdots3}{1\cdots3}$ and $C_1 = \binom{3\cdots5}{1\cdots3}$. Consider the case $0 < \alpha < \frac{1}{4}$ and $\beta = 4\alpha + 1$, leading to the matrix norms $\|C_0\|_\infty = \|C_1\|_\infty = \frac{2}{3}(\alpha + 1) < 1$. For these values of the parameters α and β, the SSS produces a continuous limit function, as the criterion of Theorem 3 is satisfied. $\diamondsuit$

3.4 Derivative Processes

Recall that an SSS of order $2m$ generates a sequence of C^{m-1} spline functions $X^{(k)}(t)$ of order $2m$. We will derive criteria for a C^r limit function f, $r = 1, \ldots, m-1$, by studying the first $m-1$ derivatives of these spline functions. Clearly, the r–th derivative $(d^r/dt^r)X^{(k)}(t)$ is a C^{m-1-r} spline function with the knot vector

$$\mathcal{T}^{(k)} = (\{t_{-1}^{(k)}\}^{\langle m-r\rangle}, \underbrace{\{t_0^{(k)}\}^{\langle m\rangle}, \ldots, \{t_{2^k n+n_1}^{(k)}\}^{\langle m\rangle}}_{\text{inner knots have multiplicity } m}, \{t_{2^k n+n_1+1}^{(k)}\}^{\langle m-r\rangle}).$$

Its control points are associated with Greville abscissas which divide the real axis according to the ratios

$$\ldots : 1 : \underbrace{2 : 2 : \ldots : 2}_{m-1-r \text{ times}} : \underbrace{1 : 1 : \ldots : 1}_{r+1 \text{ times}} : \underbrace{2 : 2 : \ldots : 2}_{m-1-r \text{ times}} : 1 : \ldots \tag{39}$$

Note that this is again a non–uniform parameterization, except for the last derivative, where $r = m - 1$.

First derivative. Consider the first derivative of the spline functions. If the conditions (29),(30) are satisfied, then a short calculation confirms that the control points of the first derivatives are again generated by a subdivision scheme $\partial\mathcal{S}$ with a certain generator matrix $D^{[1]}$ (see below).

Again, we have two necessary condition for convergence of the derivative scheme $\partial\mathcal{S}$ to a non–zero continuous limit function for arbitrary initial data.

Firstly, the rows of the generator matrix $D^{[1]}$ have to sum up to one. That is, the scheme $\partial\mathcal{S}$ has to reproduce constant functions. Equivalently, the original scheme $\mathcal{S}$ has to reproduce linear polynomials.

Secondly, the submatrices governing the transformation of the control point vectors have to have a common row, i.e.,

$$M - (m+1) \geq 1 + m \quad \text{or, equivalently,} \quad m(n_1 + 1) \geq 2. \tag{40}$$

Higher order derivatives. We apply this idea successively to the first $m-1$ derivatives. This leads to the subdivision schemes

$$\partial^r\mathcal{S}(\hat{A}_0, \ldots, \hat{A}_l, \hat{B}_0, \ldots, \hat{B}_l), \quad r = 1, \ldots, m-1 \tag{41}$$

with associated generator matrices $D^{[r]}$ (see below).

Again, we have two necessary conditions for convergence of the r–th derivative to a continuous non–zero limit function for arbitrary initial data. Firstly, the rows of its generator matrix $D^{[r]}$ have to sum up to one. Equivalently, the original scheme $\mathcal{S}$ has to reproduce the control points of polynomials of degree r. Secondly, the two submatrices $D_0^{[r]}$ and $D_1^{[r]}$ have to have a common row,

$$M - m - r \geq 1 + m, \quad \text{or, equivalently,} \quad m(n_1 + 1) \geq 1 + r. \tag{42}$$

Generator matrices. Assume that the SSS and the associated $m-1$ derivative schemes satisfy these necessary conditions. Then, the r–th derivative process $\partial^r\mathcal{S}$ has the $(M-r)\times(M-r)$ generator matrix

$$D^{[r]} = 2^r E_M^{[r]}\, G\, \left(E_M^{[r]}\right)^{-1} \binom{1,\ldots,M-r}{1,\ldots,M-r}, \quad r = 1,\ldots,m-1, \tag{43}$$

where

$$E_M^{[r]} = \mathrm{diag}(\underbrace{1,\ldots,1}_{m-r \text{ times}}, \underbrace{2,2^2,\ldots,2^r}_{r \text{ times}}, \underbrace{1,\ldots,1}_{m-r \text{ times}},\ldots,\underbrace{2,2^2,\ldots,2^r}_{r \text{ times}}) \cdot E_M, \tag{44}$$

see (35). The control points of the r–th derivative of the spline function $X^{(k)}(t)$ will be denoted by $p_i^{(k)[r]}$. Again, the control points governing the future behaviour of the scheme $\partial^r\mathcal{S}$ in the interval $[t_i^{(k)}, t_{i+1}^{(k)}]$ are gathered in a control point vector $\mathbf{p}_i^{(k)[r]} = (p_{mi}^{(k)[r]},\ldots,p_{m(i+n_1+2)-1-r}^{(k)[r]})$. The transformation of these vectors is again given by submatrices of the generator matrix,

$$\mathbf{p}_{2i}^{(k+1)[r]} = D_0^{[r]}\mathbf{p}_i^{(k)[r]} \quad \text{and} \quad \mathbf{p}_{2i+1}^{(k+1)[r]} = D_1^{[r]}\mathbf{p}_i^{(k)[r]}, \tag{45}$$

where $D_0^{[r]} = \binom{1\cdots M-m-r}{1\cdots M-m-r}$ and $D_1^{[r]} = \binom{1+m\cdots M-r}{1\cdots M-m-r}$.

Example. We consider again the case of an SSS of order $2m = 4$, generating a sequence of cubic spline functions. Here, the sequence of the first $m-1$ derivatives reduces to the first derivative. It is generated by the first derivative scheme $\partial\mathcal{S}$ producing a sequence of quadratic spline functions with double knots, see Figure 3.

We compute the generator matrix of this scheme in the case of Merrien's HSS and the associated SSS, cf. (43). This leads to

$$D^{[1]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{-3+36\alpha+\beta}{6} & \frac{3-\beta}{3} & \frac{3-36\alpha+\beta}{6} & 0 & 0 \\ \frac{3-\beta}{6} & \frac{\beta}{3} & \frac{3-\beta}{6} & 0 & 0 \\ \frac{3-36\alpha+\beta}{6} & \frac{3-\beta}{3} & \frac{-3+36\alpha+\beta}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{46}$$

The transformation of the control point vectors $\mathbf{p}_i^{(k)[r]}$ (length 3) is governed by the submatrices $D_0^{[1]} = D^{[1]}\binom{1\cdots 3}{1\cdots 3}$ and $D_1^{[1]} = D^{[1]}\binom{3\cdots 5}{1\cdots 3}$. ◊

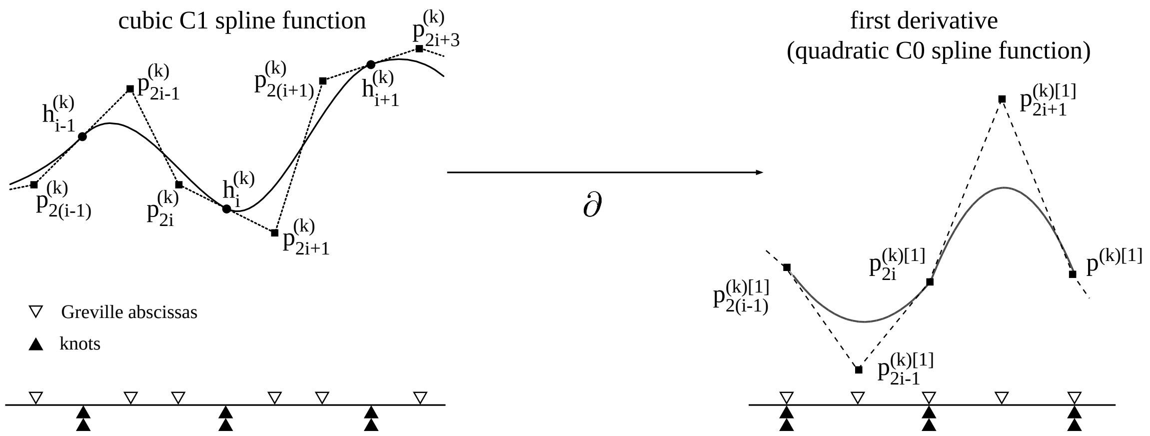

Figure 3: HSS of order $m = 2$, SSS of order $2m = 4$ (left), and the associated first derivative scheme (right).

3.5 Analyzing the derivative processes

In order to analyze the convergence of the subdivision schemes (41) we have to consider the associated difference schemes

$$\Delta\partial^r \mathcal{S}(\hat{A}_0, \ldots, \hat{A}_l, \hat{B}_0, \ldots, \hat{B}_l), \quad r = 1, \ldots, m-1, \tag{47}$$

generating the differences $\Delta_i^{(k)[r]} = p_{i+1}^{(k)[r]} - p_i^{(k)[r]}$ of the control points. The difference scheme exists if the derivative scheme $\partial^r \mathcal{S}$ satisfies the necessary conditions for convergence, i.e., the rows of the generator matrix $D^{[r]}$ sum up to one. The difference scheme $\Delta\partial^r \mathcal{S}$ has the generator matrix

$$C^{[r]} = E_{M-r} D^{[r]} \left(E_{M-r}\right)^{-1} \begin{pmatrix} 1 & \cdots & M-r-1 \\ 1 & \cdots & M-r-1 \end{pmatrix}. \tag{48}$$

The future behaviour of the process $\Delta\partial\mathcal{S}$ in the interval $[t_i^{(k)}, t_{i+1}^{(k)}]$ is determined by the difference vectors $\mathbf{\Delta}_i^{(k)[r]} = (\Delta_{mi}^{(k)[r]}, \ldots, \Delta_{m(i+n_1+2)-2-r}^{(k)[r]})^\top$ Again, the evolution of these difference vectors is governed by two submatrices of the generator matrix, $C_0^{[r]} = C^{[r]}\binom{1 \cdots M-m-r-1}{1 \cdots M-m-r-1}$ and $C_1^{[r]} = C^{[r]}\binom{1+m \cdots M-r-1}{1 \cdots M-m-r-1}$.

The convergence of the derivative schemes (41) can be analyzed with the help of the following result.

Theorem 4 (Differentiability I) *Let the subdivision process (41) satisfy the necessary conditions for convergence. Then the following are equivalent:*

1. *The derivative scheme $\partial^r \mathcal{S}$ converges uniformly to a continuous limit function $f^{[r]}$ on $[0, n]$ for arbitrary initial data.*

2. *The difference process $\Delta\partial^r \mathcal{S}$ converges uniformly to zero on $[0, n]$ for arbitrary initial data.*

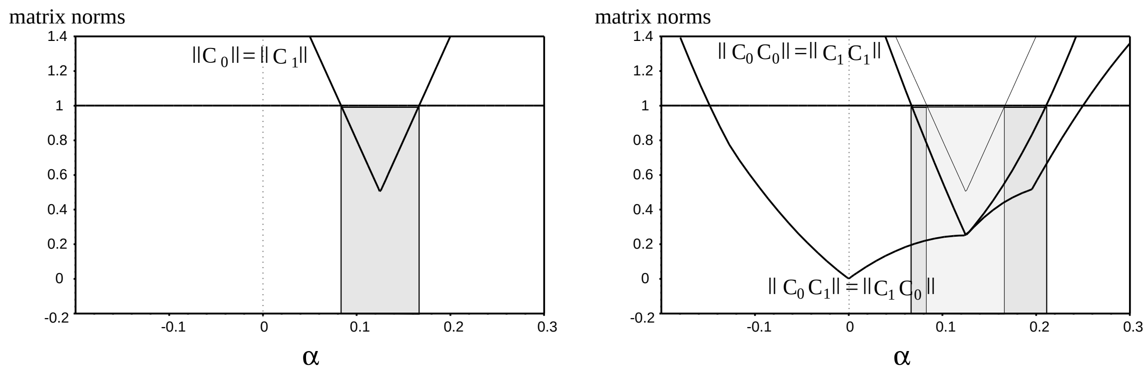

Figure 4: C^1 convergence regions of Merrien's HSS, obtained by considering one (left) or two (right) steps of the difference scheme $\Delta\partial\mathcal{S}$.

3. There exists a positive integer L and an $\alpha \in [0,1)$, such that

$$\|C_{i_1}^{[r]} \cdots C_{i_L}^{[r]}\|_\infty \leq \alpha \quad \textit{for all } i_j \in \{0,1\},\ j = 1,\ldots,L. \tag{49}$$

If these equivalent conditions are satisfied, then the original SSS produces a C^r limit function f.

Example. We apply this technique to Merrien's HSS. The generator matrix $D^{[1]}$ of the derivative scheme satisfy the necessary condition for convergence; its rows sum up to one. Consequently, we may compute the generator matrix of the associated difference process $\Delta\partial\mathcal{S}$,

$$C^{[1]} = \begin{pmatrix} \frac{9-36\alpha-\beta}{6} & \frac{3-36\alpha+\beta}{6} & 0 & 0 \\ \frac{-3+\beta+18\alpha}{3} & \frac{18\alpha-\beta}{3} & 0 & 0 \\ \frac{18\alpha-\beta}{3} & \frac{-3+\beta+18\alpha}{3} & 0 & 0 \\ \frac{3-36\alpha+\beta}{6} & \frac{9-36\alpha-\beta}{6} & 0 & 0 \end{pmatrix}. \tag{50}$$

The transformation of the difference vectors $\mathbf{\Delta}_i^{(k)[r]}$ is governed by the two submatrices $C_0^{[1]} = C^{[1]}\binom{1\cdots2}{1\cdots2}$ and $C_1^{[1]} = C^{[1]}\binom{3\cdots4}{1\cdots2}$.

Consider again the special case $\beta = 4\alpha + 1$. In order to apply the convergence criterion of Theorem 4 we have generated plots of the matrix norms in (49) for $L = 1$ (left) and $L = 2$ (right), depending on the parameter α. If all matrix norms are less than 1, then the scheme is guaranteed to produce a C^1 limit function. From (49) with $L = 1$ we get that the scheme is C^1 for all $\alpha \in [0.084, 0.166]$ (left figure). Considering two steps, i.e. choosing $L = 2$, we may conclude that the scheme is C^1 for all $\alpha \in [0.067, 0.210]$ (right figure). By composing more than two steps (i.e., increasing the value of L), we may obtain even larger intervals for α.

Based on a different approach, Merrien's scheme has been shown to produce a C^1 limit curve for $0 < \alpha < \frac{1}{4}$, see Dyn and Levin 1995. $\Diamond$

3.6 Inscribed Polygon Process

The derivative $(d^{m-1}/dt^{m-1})X^{(k)}$ is a C^0 spline curve; it is described as a sequence of Bézier curves of degree m with identical boundary control points (or, equivalently, as a spline curve of order $m+1$, where all knots have multiplicity m). Generally, the derivatives of order m of the spline functions $X^{(k)}(t)$ are not continuous. In order to analyze the differentiability of higher order we introduce the *inscribed polygon process*

$$\mathcal{P}\partial^{m-1}\mathcal{S}(\hat{A}_0, \ldots, \hat{A}_l, \hat{B}_0, \ldots, \hat{B}_l). \tag{51}$$

This process generates the sequence of piecewise linear function with the vertices (depending on the subdivision level k)

$$\left(\frac{i}{m}2^{-(k+1)}, \frac{d^{m-1}}{dt^{m-1}}X^{(k)}(\frac{i}{m}2^{-(k+1)}) \right)_{i=0,\ldots}. \tag{52}$$

That is, the process generates an inscribed polygon to the derivatives of order $m-1$, with the stepsize $1/(2^{(k+1)}m)$.

After some computations we arrive at the $(M-m+1 \times M-m+1)$ generator matrix of the inscribed polygon of the $(m-1)$–th derivative.

$$P = L_{M-m+1} \, D^{[m-1]} \, (L_{M-m+1})^{-1}, \tag{53}$$

where the transformation matrix is a bandmatrix with bandwidth $2m-1$,

$$L_{M-m+1} = \begin{pmatrix} R_{m+1} & & & & 0 \\ & \boxed{1} & & & \\ & R_{m+1} & & & \\ & & \ddots & & \\ & & & R_{m+1} & \\ & & & \boxed{1} & \\ 0 & & & & R_{m+1} \end{pmatrix}, \tag{54}$$

with

$$R_{m+1} \quad = \quad \left(\binom{m}{j-1} \left(\frac{m+1-i}{m}\right)^{m-j+1} \left(\frac{i-1}{m}\right)^{j-1} \right)_{i,j=1,\ldots,m+1}. \tag{55}$$

The inverse matrix of (53) is obtained by replacing the submatrices on the diagonal with the inverse matrices R_{m+1}^{-1}.

The inscribed polygon process and the $(m-1)$–th derivative process can be shown to be equivalent.

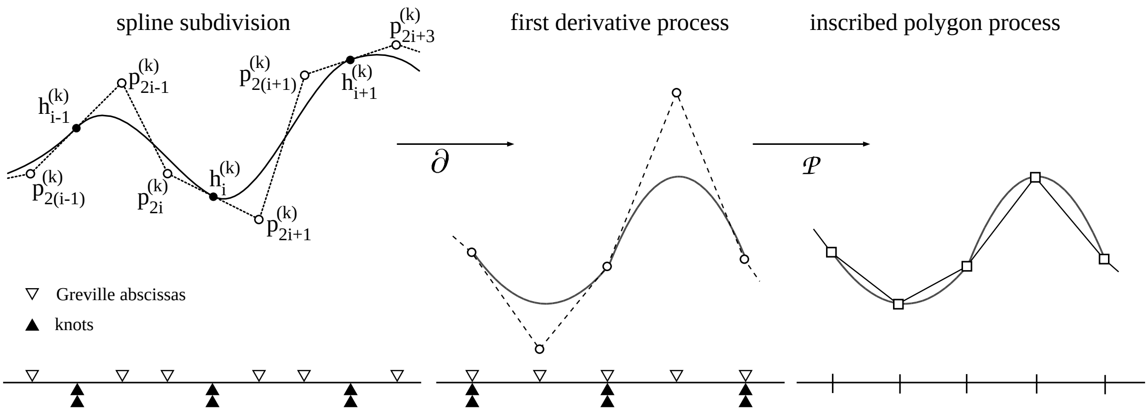

Figure 5: Spline subdivision scheme, derivative scheme, and inscribed polygon process for $m = 2$.

Theorem 5 *The inscribed polygon process $\mathcal{P}\partial^{m-1}\mathcal{S}$ generates a continuous limit function f if and only if the $(m-1)$-th derivative process $\mathcal{P}\partial^{m-1}\mathcal{S}$ generates a continuous limit function g. Moreover $f = g$.*

The proof follows directly from the stability of the Bernstein basis, and from the convex hull property of Bézier curves.

Example. In the case of Merrien's HSS we obtain the matrices

$$L_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{56}$$

which transform the matrix $D^{[1]}$ (see (46)) into the generator matrix of the inscribed polygon process $\mathcal{P}\partial\mathcal{S}$,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{-3+72\alpha+2\beta}{24} & \frac{6-\beta}{6} & \frac{3-72\alpha+2\beta}{24} & 0 & 0 \\ \frac{3-2\beta}{6} & \frac{2\beta}{3} & \frac{3-2\beta}{6} & 0 & 0 \\ \frac{3-72\alpha+2\beta}{24} & \frac{6-\beta}{6} & \frac{-3+72\alpha+2\beta}{24} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{57}$$

Figure 5 shows the spline subdivision scheme $\mathcal{S}$, the derivative scheme $\partial\mathcal{S}$, and the associated inscribed polygon process $\mathcal{P}\partial\mathcal{S}$. ◊

3.7 Analyzing the inscribed polygon process

The limit curve of the inscribed polygon process can be analyzed as in the classical case of point subdivision schemes, using divided difference processes and their difference schemes, see Dyn et al. 1991. We denote by

$$\mathcal{D}^r\mathcal{P}\partial^{m-1}\mathcal{S}(\hat{A}_0,\dots,\hat{A}_l,\hat{B}_0,\dots,\hat{B}_l). \tag{58}$$

the r–th divided difference process of the inscribed polygon scheme, and by

$$\Delta\mathcal{D}^r\mathcal{P}\partial^{m-1}\mathcal{S}(\hat{A}_0,\dots,\hat{A}_l,\hat{B}_0,\dots,\hat{B}_l). \tag{59}$$

its difference process, $r=0,1,\dots$.

If the r–th divided difference process converges to a smooth limit function for arbitrary initial data, then two necessary conditions are always satisfied. Firstly, the rows of its generator matrix sum up to one. Consequently, the inscribed polygon process (58) has to reproduce polynomials of degree r. Secondly, the submatrices governing the transformation of the control point vectors have to overlap,

$$M-2m-(r-1)\geq(1+m),\quad\text{or, equivalently,}\quad mn_1\geq r. \tag{60}$$

Let $Q^{[r]}$ and $E^{[r]}$ be the generator matrix of the r–th divided difference process (58), and of its difference process (59). Starting from the generator matrix $Q^{[0]}=P$ of the inscribed polygon process, these matrices can recursively be obtained from

$$Q^{[r+1]}=2\,E_{M-m+1-r}Q^{[r]}(E_{M-m+1-r})^{-1}\begin{pmatrix}1&\cdots&M-m-r\\1&\cdots&M-m-r\end{pmatrix},\quad r=0,1,\dots, \tag{61}$$

and $E^{[r]}=\frac{1}{2}Q^{[r+1]}$. The transformation of the corresponding difference vectors is then governed by the two submatrices $E_0^{[r]}=E^{[r]}\binom{1\,\cdots\,M-2m-r}{1\cdots M-2m-r}$ and $E_1^{[r]}=E^{[r]}\binom{1+m\,\cdots\,M-m-r}{1\cdots M-2m-r}$. With the help of these matrices we obtain a criterion for differentiability of the limit function of the SSS.

Theorem 6 (Differentiability II) *Assume that the inscribed polygon process and the associated divided difference processes satisfy the necessary conditions. Then the following are equivalent:*

1. *The scheme $\mathcal{D}^r\mathcal{P}\partial^{m-1}\mathcal{S}$ converges uniformly to a continuous limit function on $[0,n]$ for arbitrary initial data.*

2. *The difference process $\Delta\mathcal{D}^r\mathcal{P}\partial^{m-1}\mathcal{S}$ converges uniformly to zero on $[0,n]$ for arbitrary initial data.*

3. *There exists an integer $L>0$ and an $\alpha\in[0,1)$, such that*

$$\|E_{i_1}^{[r]}\cdots E_{i_L}^{[r]}\|_\infty\leq\alpha\quad\forall\ i_j\in\{0,1\},\ j=1,\dots,L. \tag{62}$$

If these equivalent conditions are satisfied, then the inscribed polygon process produces a C^r limit function, and the original SSS produces a C^{m+r-1} limit function.

Remarks. 1.) Note that this theorem – for $r = 0$ – provides another criterion for differentiability of order $m-1$. That is, in order to make sure that the original SSS generates a C^{m-1} function, it is sufficient to show that one of the processes $\Delta\partial^{m-1}\mathcal{S}$ and $\Delta\mathcal{P}\partial^{m-1}\mathcal{S}$ is contracting.

2.) Note that the differences and divided differences are generated by subdivision schemes only if the necessary conditions are satisfied, cf. the remarks after Eq. (35).

Example. At first we analyze the continuity of the limit function of the inscribed polygon process. This leads again to criteria for a C^1 limit function of the Merrien's HSS and the associated SSS.

From (57) we obtain the generator matrix of the difference process $\Delta\mathcal{P}\partial\mathcal{S}$,

$$E^{[0]} = \begin{pmatrix} \frac{27-72\alpha-2\beta}{24} & \frac{3-72\alpha+2\beta}{24} & 0 & 0 \\ \frac{-15+72\alpha+10\beta}{24} & \frac{9+72\alpha-10\beta}{24} & 0 & 0 \\ \frac{9+72\alpha-10\beta}{24} & \frac{-15+72\alpha+10\beta}{24} & 0 & 0 \\ \frac{3-72\alpha+2\beta}{24} & \frac{27-72\alpha-2\beta}{24} & 0 & 0 \end{pmatrix}. \tag{63}$$

The transformation of the difference vectors is governed by the two submatrices $E_0^{[0]} = E^{[0]}\binom{1\cdots2}{1\cdots2}$ and $E_1^{[0]} = E^{[0]}\binom{3\cdots4}{1\cdots2}$. Again we consider the matrix norms for $L = 2$, where $\beta = 4\alpha + 1$ has been chosen. This leads to the feasible domain $\alpha \in [0.030, 0.235]$, where the inscribed polygon scheme is guaranteed to produce a continuous limit function. Consequently, for these values of α, the spline curves generated by the SSS converge to a C^1 limit function. The feasible interval obtained by considering two steps of the difference process $\Delta\mathcal{P}\partial\mathcal{S}$ is slightly larger than the analogous interval obtained from the (equivalent) difference process $\Delta\partial\mathcal{S}$. By composing more and more steps, however, the results obtained from both processes will become more and more similar.

Secondly we consider the divided difference scheme $\mathcal{D}\mathcal{P}\partial\mathcal{S}$, with the generator matrix $Q^{[1]} = 2E^{[0]}$. We check the two necessary conditions for convergence to a continuous limit function for arbitrary initial data.

According to the first condition, the rows of its generator matrix have to sum up to one. Equivalently, the scheme $\mathcal{P}\partial\mathcal{S}$ has to reproduce linear functions. This necessary condition implies $\alpha = \frac{1}{8}$. The second condition, however, is violated, as $mn_1 = 2 \cdot 0 < 1$. Thus, Merrien's scheme is not C^2. $\Diamond$

4 Design

The techniques presented in this paper can be used in order to design HSS and SSS. We present two examples.

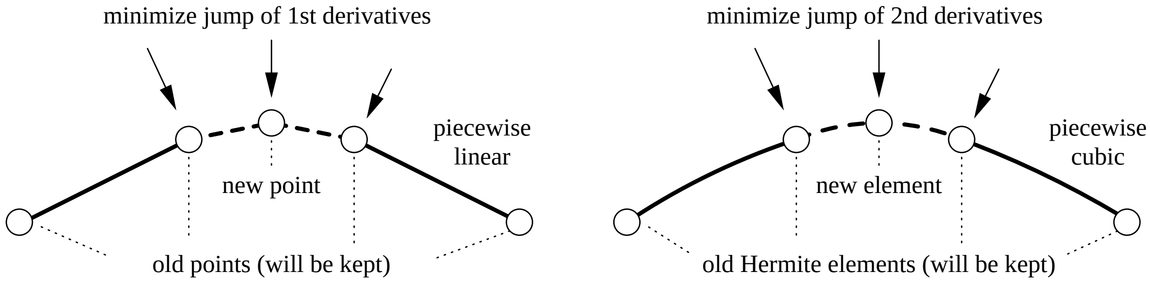

Figure 6: Deriving the interpolatory 4–point scheme (left) and the analogous scheme for second order Hermite elements (right).

4.1 A Generalization of the 4–Point scheme

Based on a geometric construction, Dyn et al. (1987) derived a family of interpolating 4-point subdivision schemes. With the help of the spline–based approach, this family can be generalized to Hermite elements (cf. Schwanecke and Jüttler 1999), as follows. Consider the piecewise linear function with vertices $(t_i^{(k)}, \mathbf{h}_i^{(k)})_{i\in\mathbb{Z}}$. This function is refined by inserting new vertices $(t_{2i+1}^{(k+1)}, \mathbf{h}_{2i+1}^{(k+1)})_{i\in\mathbb{Z}}$ and keeping the old vertices, $\mathbf{h}_{2i}^{(k+1)} = \mathbf{h}_i^{(k)}$. The new vertices are placed by minimizing the jumps of the first derivatives at the segment end points, see Figure 6. More precisely, the weighted sum of squared differences, with weights 2ω, $(1-4\omega)/4$, and 2ω is to be minimized by the new vertex. This idea leads to the refinement rules of the original 4–point scheme.

Analogously, one may consider the piecewise cubic function, interpolating the second order Hermite elements $(t_i^{(k)}, \mathbf{h}_i^{(k)})_{i\in\mathbb{Z}}$. This function is refined by inserting new Hermite elements $(t_{2i+1}^{(k+1)}, \mathbf{h}_{2i+1}^{(k+1)})_{i\in\mathbb{Z}}$ and keeping the old ones, $\mathbf{h}_{2i}^{(k+1)} = \mathbf{h}_i^{(k)}$. The new Hermite elements are placed by minimizing the jumps of the second derivatives at the segment end points, see again Figure 6. More precisely, the weighted sum of squared differences, with weights 8ω, $(1-8\omega)/8$, and 8ω is to be minimized by the new Hermite element. This leads to the refinement rules of the generalized 4–point scheme. The associated SSS of order $2m = 4$ with $l = 3$ has the matrix masks

$$\begin{gathered}\hat{A}_0 = \hat{A}_2 = \hat{A}_3 = 0_{2\times 2},\ \hat{A}_1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix},\ \hat{B}_0 = \begin{pmatrix} 0 & \frac{1}{8}+\omega \\ 0 & \frac{1}{8}-\omega \end{pmatrix}, \\ \hat{B}_1 = \begin{pmatrix} -\frac{1}{8}-2\omega & \frac{3}{4}+3\omega \\ -\frac{1}{4}+2\omega & \frac{3}{8}-3\omega \end{pmatrix},\ \hat{B}_2 = \begin{pmatrix} \frac{3}{8}-3\omega & -\frac{1}{4}+2\omega \\ \frac{3}{4}+3\omega & -\frac{1}{8}-2\omega \end{pmatrix},\ \hat{B}_3 = \begin{pmatrix} \frac{1}{8}-\omega & 0 \\ \frac{1}{8}+\omega & 0 \end{pmatrix}\end{gathered} \tag{64}$$

which depend on one parameter ω, see 14. The convergence of this scheme can be analyzed with the techniques described in this paper. By composing 6 steps of the difference scheme $\Delta\mathcal{D}\mathcal{P}\partial\mathcal{S}$ we obtain the C^2 convergence range $\omega \in [-0.006, 0.036]$.

Figure 7 shows several curves which have been generated by Merrien's scheme and by the generalized 4–point scheme. This scheme has been described earlier in

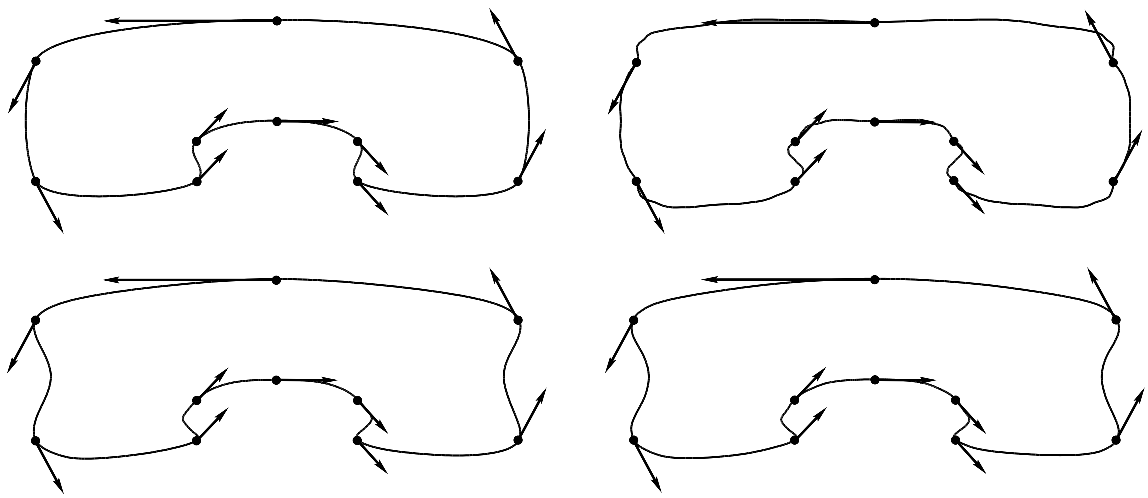

Figure 7: Top: Merrien's interpolants for $\alpha = 0.125$ (left) and $\alpha = 0.2$ (right). Below: Interpolants generated by the generalized 4–point–scheme, where $\omega = 0.01$ (left) and $\omega = 0.03$ (right).

a conference article (Schwanecke and Jüttler 1999). See also Schwanecke 2000 for additional information.

4.2 Smoothing Scheme

Consider the sequence of Hermite elements $\mathbf{h}_i^{(k)}$ of order $m = 2$, associated with the dyadic points $t_i^{(k)}$. We fit cubic polynomials $F_i(t)$ and $G_i(t)$ to neighbouring elements, by minimizing the weighted sums

$$\sum_{j=-K}^{K} v_j \, (F_i(t_{i+j}^{(k)}) - h_{i+j,0}^{(k)})^2 + \frac{\lambda}{4^k} \sum_{j=-K}^{K} v_j \, (F_i'(t_{i+j}^{(k)}) - h_{i+j,1}^{(k)})^2 \qquad (65)$$

and

$$\sum_{j=-L}^{L+1} w_j \, (G_i(t_{i+j}^{(k)}) - h_{i+j,0}^{(k)})^2 + \frac{\lambda}{4^k} \sum_{j=-L}^{L+1} w_j \, (G_i'(t_{i+j}^{(k)}) - h_{i+j,1}^{(k)})^2 \qquad (66)$$

with weights $v = (v_{-K}, \ldots, v_K)$, $w = (w_{-L}, \ldots, w_{L+1})$ and λ. These polynomials depend on $2K + 1$ resp. $2L + 2$ Hermite elements. The new Hermite elements are sampled from the resulting cubic polynomials,

$$\mathbf{h}_{2i}^{(k+1)} = (F_i(t_i^{(k)}), F_i'(t_{2i}^{(k+1)})), \quad \mathbf{h}_{2i+1}^{(k+1)} = (G_i(t_i^{(k)}), F_i'(t_{2i+1}^{(k+1)})). \qquad (67)$$

This leads to a family of stationary (due the choice of the weight $\lambda/4^k$ of the first derivative) Hermite subdivision schemes. In order to apply the previous theory, one needs to apply a suitable shift of indices. Two examples will be discussed in more

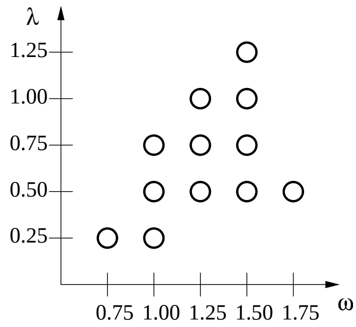

Figure 8: Estimated C^2 convergence range of the first smoothing scheme.

detail.

Example 1. We choose $K = 2$, $L = 1$, and the weights $v = (1, \omega, \omega, \omega, 1)$, $w = (1, \omega, \omega, 1)$. This produces a family of HSS and associated SSS with $l = 4$, depending on the two parameters λ and ω. The parameter λ controls the influence of the first derivative, and the parameter ω can be used to adjust the influence of the boundary points in the masks.

For suitable values of λ and ω, the resulting scheme can be shown to produce a C^2 limit function. It is, however, difficult to compute the matrix norms in Theorem 6 exactly, as this involves norms of 13×13 matrices whose components are fairly complicated functions of ω and λ. In order to estimate the C^2 convergence range we computed these matrix norms numerically for pairs (ω, λ) from a grid with stepsize 0.25. Figure 8 shows the feasible pairs, obtained by composing 3 steps of the difference process $\Delta\mathcal{DP}\partial\mathcal{S}$. Examples for limit curves obtained for two feasible values of the parameters are shown in Figure 9. We have drawn both the points and the associated derivative vectors. The derivatives have been scaled by $\frac{1}{2}$.

Example 2. Similar to the first example, we choose $K = 1$, $L = 1$, and the weights $v = (1, \omega, 1)$, $w = (1, \omega, \omega, 1)$. Once again, this produces a two–parameter family of HSS and associated SSS. The subdivision masks are smaller than in the previous example ($l = 3$).

As observed in our numerical experiments, these subdivision seem to generate C^2 limit curves, for suitable values of ω and λ (e.g., $\omega = 1.3, \lambda = 0.5$). However, we were not able to verify this fact with the help of Theorem 6, which provides only a sufficient condition. This will be a subject of further research.

The subdivision masks of the associated SSS, obtained for $\omega = 1.3, \lambda = 0.5$, are

$$\hat{A}_0 = \begin{pmatrix} -0.064 & 0.40 \\ -0.064 & 0.27 \end{pmatrix}, \hat{A}_1 = \begin{pmatrix} 0.36 & 0.14 \\ 0.14 & 0.36 \end{pmatrix}, \hat{A}_2 = \begin{pmatrix} 0.27 & -0.064 \\ 0.40 & -0.064 \end{pmatrix}, \hat{A}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{B}_0 = \begin{pmatrix} -0.12 & 0.28 \\ -0.088 & 0.19 \end{pmatrix}, \hat{B}_1 = \begin{pmatrix} 0.18 & 0.24 \\ 0.046 & 0.30 \end{pmatrix}, \hat{B}_2 = \begin{pmatrix} 0.30 & 0.046 \\ 0.24 & 0.18 \end{pmatrix}, \hat{B}_3 = \begin{pmatrix} 0.19 & -0.088 \\ 0.28 & -0.12 \end{pmatrix}.$$

Again, this scheme is illustrated by an example, see Figure 9. Note that the singu-

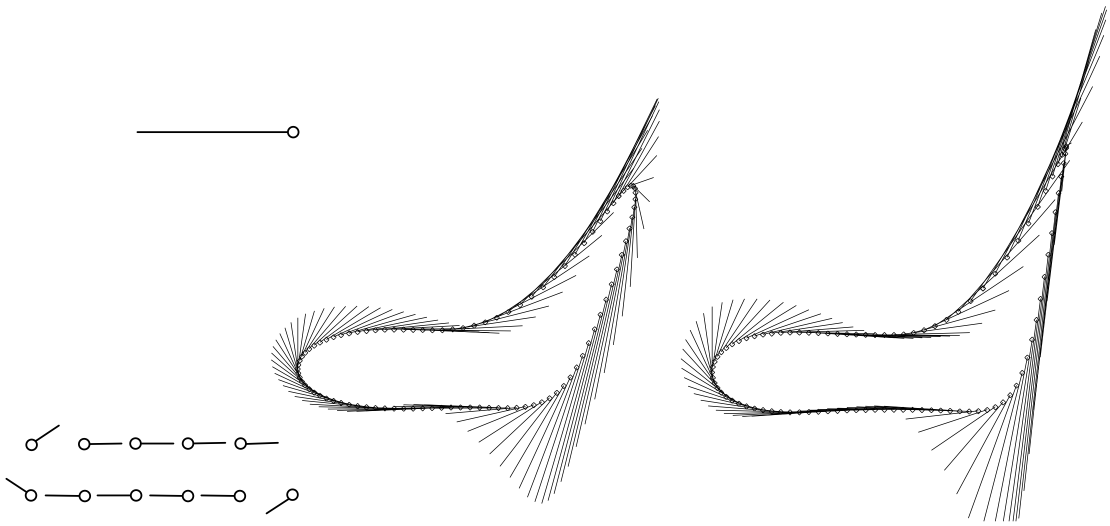

Figure 9: Sequences of Hermite elements generated by the smoothing scheme, Initial data (left, on a smaller scale), Example 1, $\omega = 1.3, \lambda = 0.6$ (center) and Example 2, $\omega = 1.3, \lambda = 0.5$ (right).

larity is due to the parameterization of the curve, not to a lack of differentiability.

5 Conclusion

We have developed a set of tools for designing Hermite subdivision schemes and for analyzing the smoothness and differentiability of the generated limit curves. Our approach was based on the interpolating spline curves which can be associated with any sequence of Hermite elements. As a matter of future research, one should try to generalize these ideas to surface design, by considering bivariate Hermite subdivision schemes, cf. Dubuc and Merrien 1999, Van Damme 1997.

Using Hermite schemes, it is possible to design simultaneously both a curve (or surface) and its tangent (or tangent plane). This may be a useful feature in various applications, where the quality of the resulting shape is important. For instance, by simultaneously designing a surface and its tangent planes, it should be relatively simple to control the distribution of reflection lines, which are a powerful tool for assessing the quality of a surface.

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List of Figures

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Bert Jüttler studied mathematics at the Universities of Technology at Dresden and Darmstadt (both in Germany). After gaining a PhD in 1994, he was a postdoctoral research assistant at Darmstadt and at the University of Dundee (Scotland), and later became lecturer at Darmstadt (since 1998). In October 2000 he was appointed as a professor of mathematics at the Johannes Kepler University of Linz, Austria. His research interests include Computer Aided Geometric Design, Kinematics and Robotics, Differential Geometry and Computer Graphics.

(Insert UliS.jpg here)

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