Assignment 1

Welcome to the first assignment of the lecture *Computer Vision*. **Please read all instructions carefully!** The goal of this assignment is a brief repetition of the most important basic math concepts needed to understand some of the theoretical background of Computer Vision.

Submission is due on Thursday, May 2nd, 2024 at 4pm via `read.mi.hs-rm.de` Please note that late assignments will receive zero (0) marks, so you are strongly encouraged to start the assignment early.

Exercise 1 (4 points)**.** Consider the three vectors

$$\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}.$$

1. Determine the *inner product* $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$.
2. Determine the *outer product* $\mathbf{x} \otimes \mathbf{y} = \mathbf{x}\mathbf{y}^\top$ of $\mathbf{x}$ and $\mathbf{y}$.
3. Determine $(\mathbf{x} \otimes \mathbf{y})\mathbf{z}$
4. What is the *rank* of $\mathbf{x} \otimes \mathbf{y}$?

Exercise 2 (4 points)**.** Consider the 2×2 matrices

$$R_1 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad R_2 = \begin{bmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix}.$$

1. Determine the determinants of the matrices.
2. Are the matrices orthogonal?
3. Are the matrices invertible? If so, give the inverse. If not, explain why.
4. What is the difference between R_1 and R_2?

Exercise 3 (4 points)**.** Show that for any nonzero vector $\mathbf{u} = (u_1, u_2, u_3)^\top \in \mathbb{R}^3$, the rank of the *skew-symmetric* matrix (sometimes also written as $\hat{\mathbf{u}}$ or $\mathbf{u}_\times$)

$$[\mathbf{u}]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

is always two. That is, the three row (or column) vectors span a two-dimensional subspace of $\mathbb{R}^3$. Hint: If we fix $\mathbf{u}$ the cross product $\mathbf{u} \times \mathbf{v}$ can be represented by linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$ represented by the matrix $[\mathbf{u}]_\times$.

Exercise 4 (4 points)**.** Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$, where $\|\mathbf{x}\| \leq 1$ with the following norms.

1. $\|\mathbf{x}\|_0 = \sum_{i=1, x_i \neq 0}^{n} 1$
2. $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$
3. $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
4. $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$

Exercise 5 (6 points)**.** What are the derivatives

1. of $y = \frac{1}{1+e^{-x}}$ with respect to x
2. of $y = |x|$ with respect to x
3. of $y = \mathbf{w}^\top \mathbf{x}$ ($\mathbf{x}, \mathbf{w} \in \mathbb{R}^n$) with respect to $\mathbf{x}$
4. of $y = \mathbf{w}^\top \mathbf{x}$ ($\mathbf{x}, \mathbf{w} \in \mathbb{R}^n$) with respect to $\mathbf{w}$
5. of $\mathbf{y} = M\mathbf{x}$ ($M \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^n$) with respect to $\mathbf{x}$
6. of y_i with respect to m_{ij}, where $\mathbf{y} = M\mathbf{x}$ ($M \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^n$)

Exercise 6 (5 points)**.** Given the data sample $S = \{0, 0, 1, 1, 1, 1\}$, created by flipping a coin x six times, where 0 denotes that the coin turned up heads and 1 denotes that it turned up tails.

1. What is the *sample mean* of S?
2. What is the *sample variance* of S?
3. What is the probability of oberving S, assuming it was generated by flipping a coin with an equal probability of heads and tail (i.e. $p(x = 0) = p(x = 1) = 0.5$)?
4. What is the probability of oberving S if $p(x = 1) = 0.6$?
5. What is the value of $p(x = 1)$ that maximizes the probability of S?

Exercise 7 (3 points)**.** Consider the following *probability table* over values x and y:

		x		
		a	b	c
y	T	0.1	0.1	0.3
	F	0.05	0.2	0.25

1. What is $p(y = T, x = b)$?
2. What is $p(y = T|\, x = b)$?
3. What is $p(y = T)$?