Computer Graphics

2D Transformations and Homogeneous Coordinates

Prof. Dr. Ulrich Schwanecke

RheinMain University of Applied Sciences

Vector Space $\mathbb{R}^n$

- Scalars $\alpha, \beta, \gamma, t \in \mathbb{R}$
    - are written in *italic* font.
- Vectors $\mathbf{x} \in \mathbb{R}^n$
    - are written in **bold** font,
    - are column vectors $\mathbf{x} \in \mathbb{R}^{n\times 1}$, and
    - their components are denoted as $\mathbf{x} = (x_1, \ldots, x_n)^\top$, which means go x_i units in the direction of the i-the basis vector of a given (Cartesian) coordinate system
- Linear combinations $\alpha\,\mathbf{x} + \beta\,\mathbf{y} \in \mathbb{R}^n$ are performed component-wise and stay in the vector space.

Euclidean Vector Space $\mathbb{R}^n$

- *Inner product* aka *dot product* aka *scalar product*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \underset{\underset{\mathbb{R}^{1\times n}}{\cap}}{\mathbf{x}}^{\top} \overset{\overset{\mathbb{R}^{n\times 1}}{\cup}}{\mathbf{y}} = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}^{1\times 1} = \mathbb{R}$$

- The induced metric measures geometric length

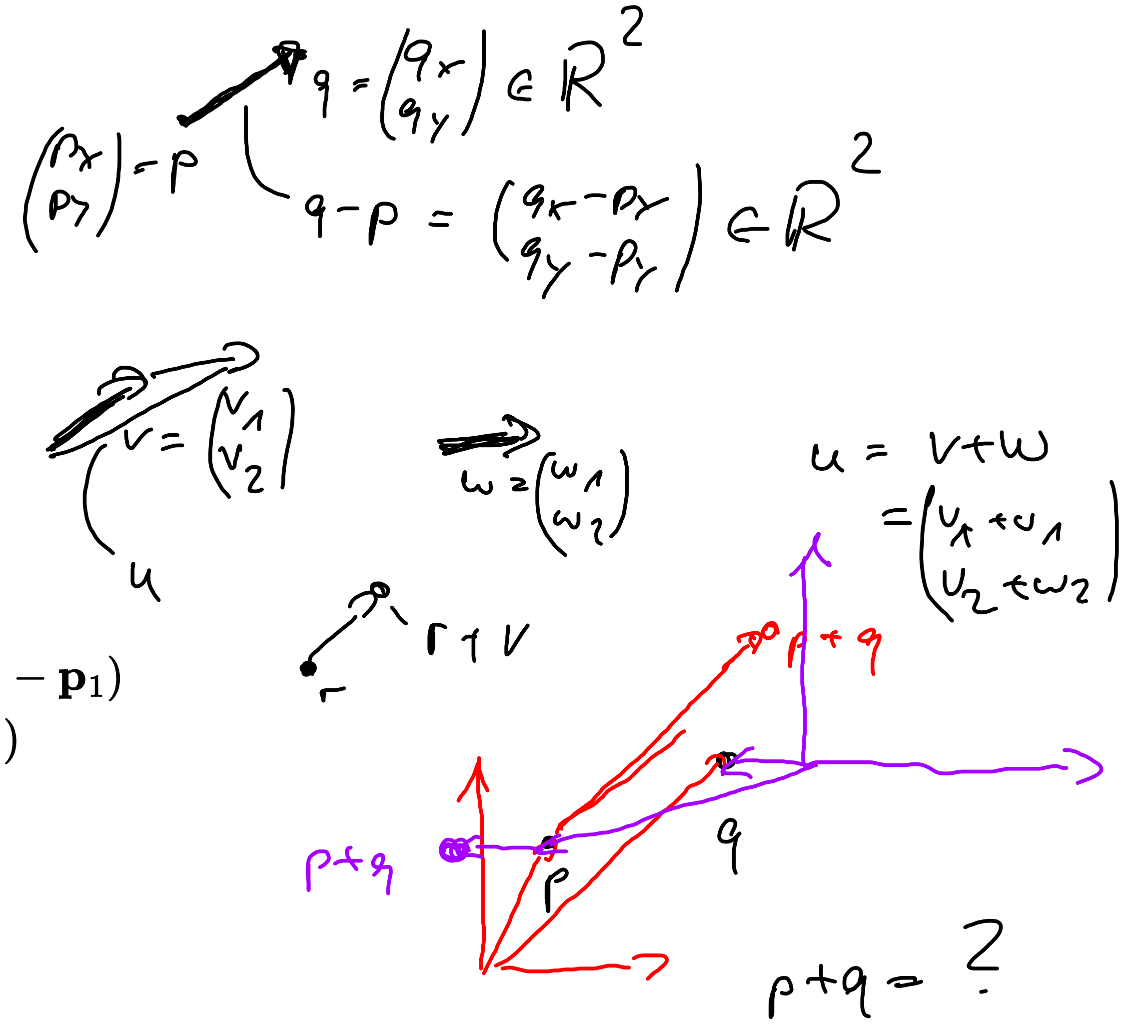
$$||\,\mathbf{x}\,|| = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

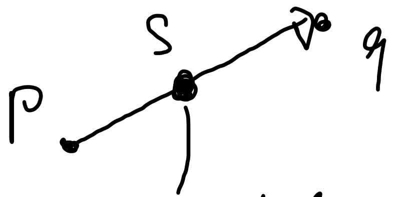
- Scalar product can measure angle θ between two vectors

$$\mathbf{x}^\top \mathbf{y} = \cos\theta \, \|\mathbf{x}\| \, \|\mathbf{y}\| \quad \Rightarrow \quad \theta = \operatorname{acos}\left(\frac{\mathbf{x}^\top \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} \right)$$

Points vs. Vectors

- Subtle distinction
  - Points denote positions in $\mathbb{R}^3$
  - Vectors denote differences of points
- Meaningful operations
  - vector + vector = vector
  - point - point = vector
  - point + vector = point
  - point + point = ???
    - **only meaningful** if $\mathbf{q} = \sum_{i=1}^{n} \alpha_i \mathbf{p}_i$ with
      - $\sum_{i=1}^{n} \alpha_i = 1 \quad \Longrightarrow \quad \mathbf{q} = \mathbf{p}_1 + \sum_{i=2}^{n} \alpha_i (\mathbf{p}_i - \mathbf{p}_1)$
      - $\sum_{i=1}^{n} \alpha_i = 0 \quad \Longrightarrow \quad \mathbf{q} = \sum_{i=2}^{n} \alpha_i (\mathbf{p}_i - \mathbf{p}_1)$





$p + \frac{1}{2}(q - p)$

$s = \frac{1}{2}p + \frac{1}{2}q$

$$\begin{array}{cc} 1p & +1q \\ \alpha_0 & \alpha_1 \\ \shortparallel & \shortparallel \\ \frac{1}{2}p & +\frac{1}{2}q \end{array}$$

$$\underline{\underline{\phantom{\tfrac{1}{2}p + \tfrac{1}{2}q}}}$$

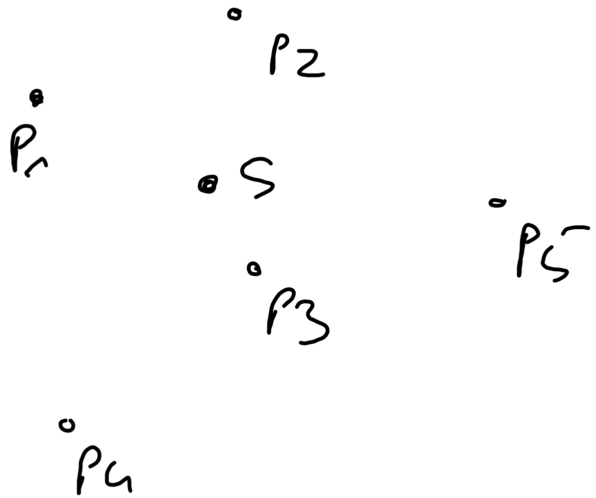
$\shortparallel$

$$1 \cdot p - \frac{1}{2}p + \frac{1}{2}q = 1 \cdot p + \frac{1}{2}(q - p)$$

$$= \underset{\substack{\uparrow \\ \text{Punkt}}}{p} + \frac{1}{2}\underset{\substack{\uparrow \\ \text{Vektor}}}{(q - p)}$$

$\alpha_0 + \alpha_1 = 2 \neq 1$

$\alpha_0 + \alpha_1 = \frac{1}{2} + \frac{1}{2} = 1$



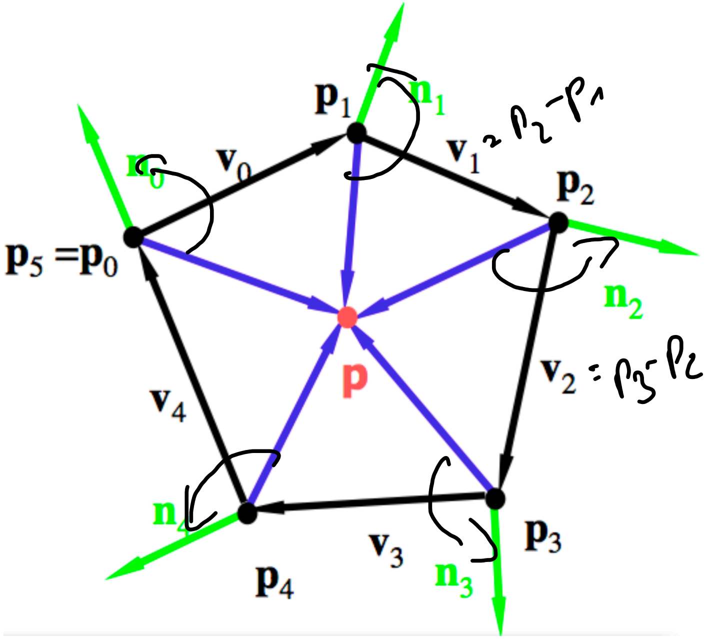
$$s = \sum_{i=1}^{5} \frac{1}{5} p_i = \frac{1}{5} \sum_{i=1}^{5} p_i$$

Application: Point in Convex Polygon

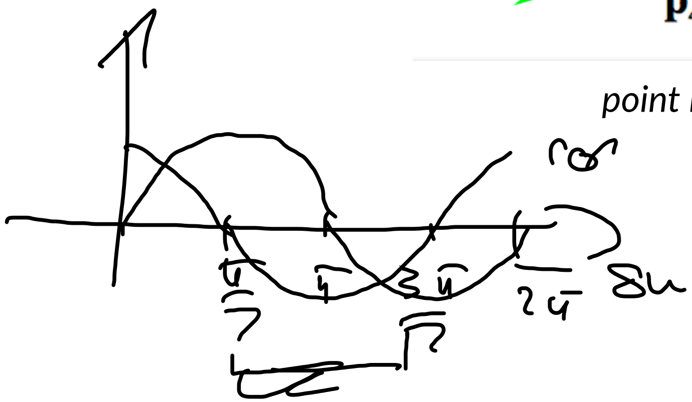
Test whether the point $\mathbf{p}$ lies in the **convex** polygon $\mathbf{p}_0, \ldots, \mathbf{p}_5 = \mathbf{p}_0$:

1. Calculate the *edges* $\mathbf{v}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$ of the polygon
2. Determine the *normal vectors* $\mathbf{n}_i$
   $[\mathbf{n}_i = (-y_i, x_i)^T$ if $\mathbf{v}_i = (x_i, y_i)^T]$
3. $\mathbf{p}$ lies inside the polygon, iff
   $\langle \mathbf{n}_i, (\mathbf{p} - \mathbf{p}_i) \rangle < 0 \quad \forall\, i = 0, \ldots, n-1$
   - if $\langle \mathbf{n}_i, (\mathbf{p} - \mathbf{p}_i) \rangle > 0$ for any i, then $\mathbf{p}$ is outside the polygon
   - $\mathbf{p}$ lies on the boundary of the polygon iff
     $\langle \mathbf{n}_i, (\mathbf{p} - \mathbf{p}_i) \rangle \leq 0 \quad \forall\, i = 0, \ldots, n-1,$
     where at least once "$=$" holds.

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} -y \\ x \end{pmatrix} = 0$$

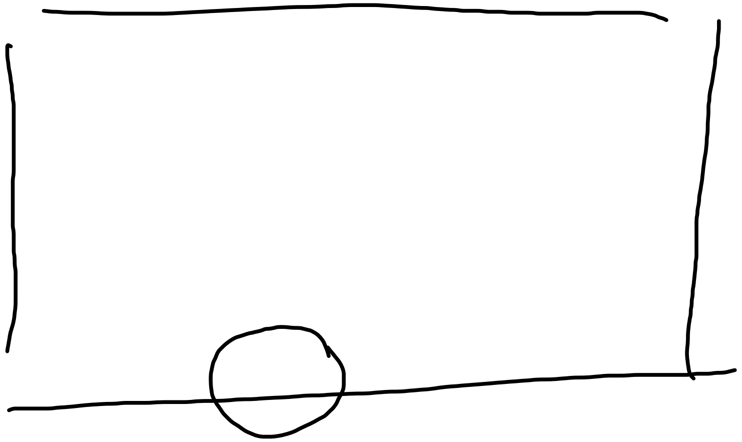


point in polygon test



Vector Product

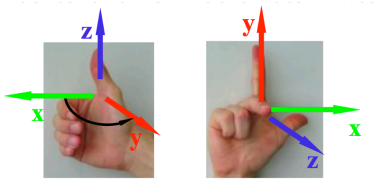
The *vector product* of the vectors $\mathbf{v}, \mathbf{w}$ is given as

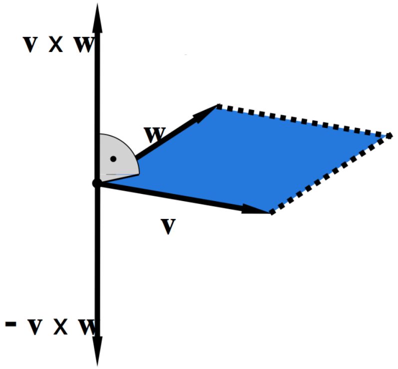


$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\times\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=\begin{vmatrix}\mathbf{e}_1&\mathbf{e}_2&\mathbf{e}_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{vmatrix}=\begin{pmatrix}v_2w_3-v_3w_2\\v_3w_1-v_1w_3\\v_1w_2-v_2w_1\end{pmatrix}=\underbrace{\begin{pmatrix}0&-v_3&v_2\\v_3&0&-v_1\\-v_2&v_1&0\end{pmatrix}}_{=:[\mathbf{v}]_\times}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$$

It holds

- $\mathbf{v}\times\mathbf{w}=[\mathbf{v}]_\times\mathbf{w}=-[\mathbf{w}]_\times\mathbf{v}=-\mathbf{w}\times\mathbf{v}$
- the vector $\mathbf{v}\times\mathbf{w}$ is **orthogonal** to the plane defined by $\mathbf{v}$ and $\mathbf{w}$
- $||\,\mathbf{v}\times\mathbf{w}\,||$ equals the **area of the parallelogram** given by $\mathbf{v}$ and $\mathbf{w}$
- if $\mathbf{v}\times\mathbf{w}=\mathbf{0}\in\mathbb{R}^3$ the vectors $\mathbf{v}$ and $\mathbf{w}$ are **collinear**

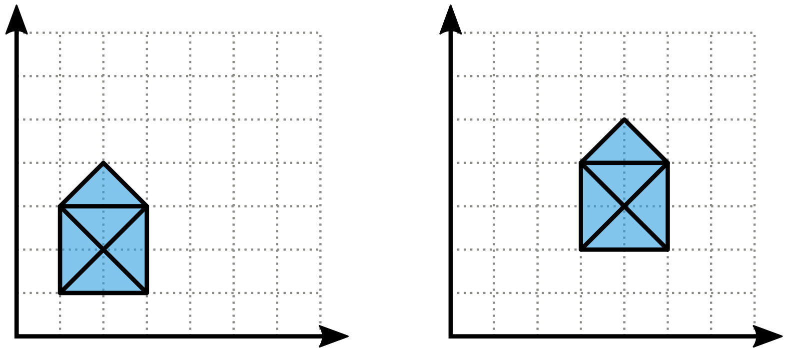




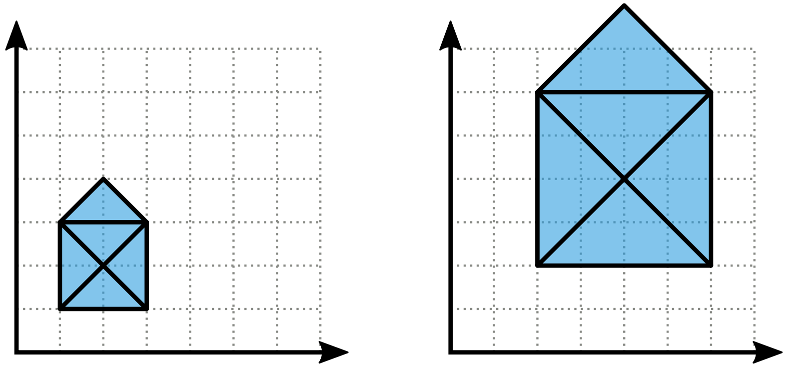
vector product

2D Transformations

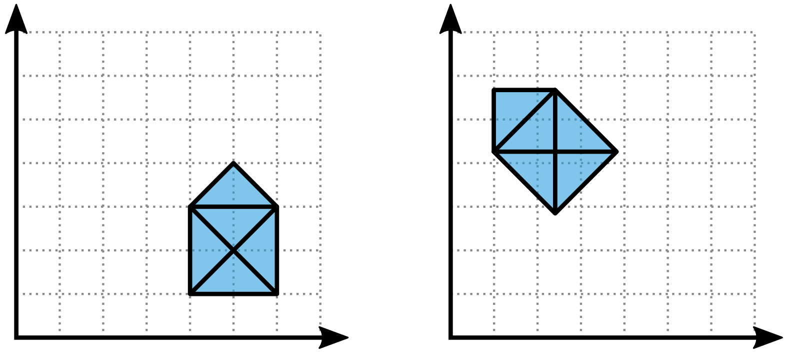
2D Transformations



translation



scaling

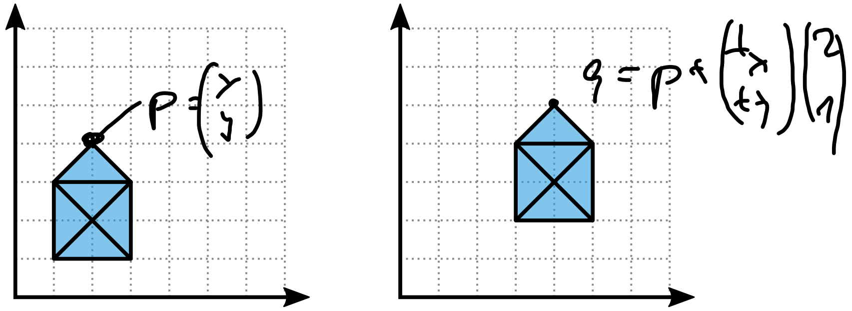


rotation

2D Translation

Translate object by t_x in x and t_y in y

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + t_x \\ y + t_y \end{pmatrix}$$

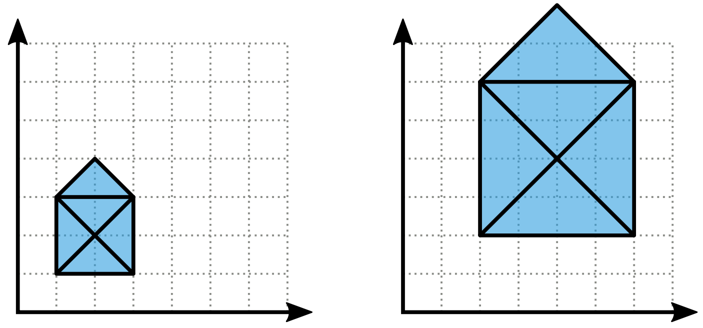


translation by $(2, 1)$

2D Scaling

Scale object by s_x in x and s_y in y
(around the origin!)

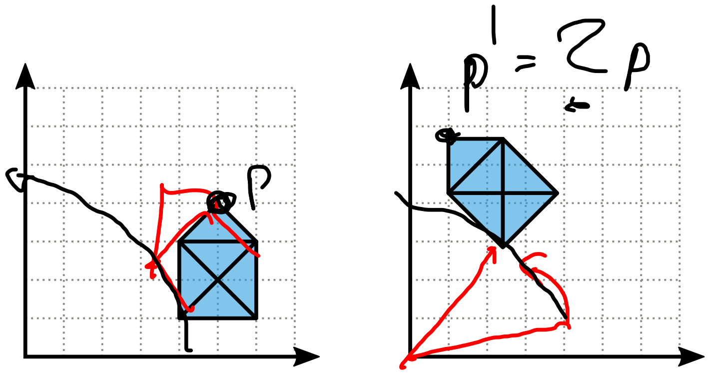
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$$



scaling by $(2, 2)$

2D Rotation

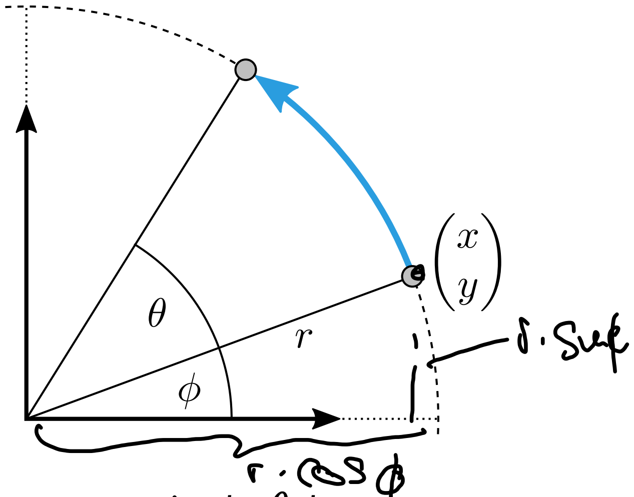
Rotate object by θ degrees
(around the origin!)



rotation by 45 degrees

2D Rotation

Rotate point $(x, y) = (r\cos\phi, r\sin\phi)$
by θ degrees around the origin



rotation by θ degrees

$$\binom{x}{y} \mapsto \begin{pmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{pmatrix}$$

$$= \begin{pmatrix} r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{pmatrix}$$

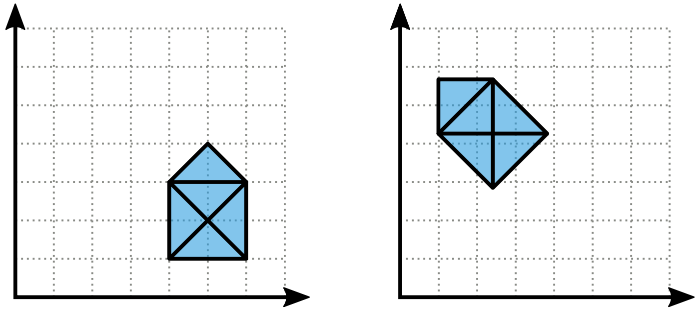
$$= \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \cos\theta \cdot y + \sin\theta \cdot x \end{pmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \binom{x}{y}$$

2D Rotation

Rotate object by θ degrees
(around the origin!)

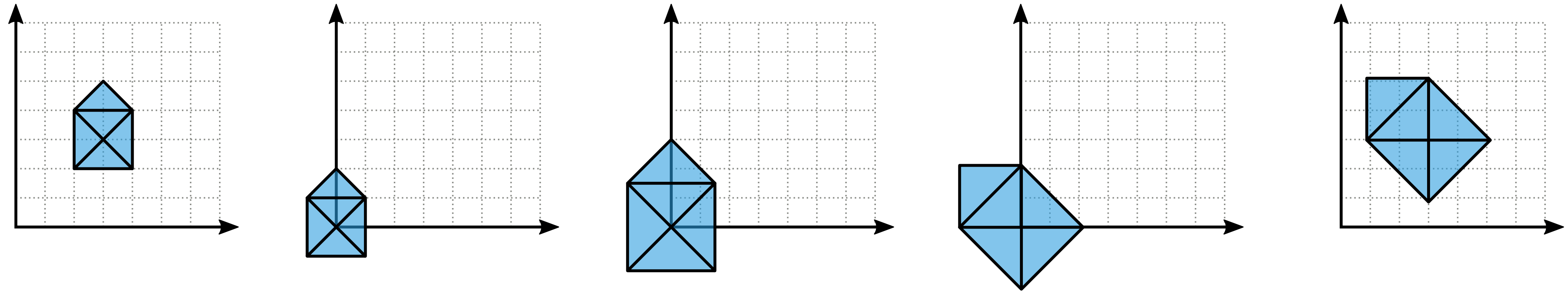
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



rotation by 45 degrees

How to rotate/scale around object center?

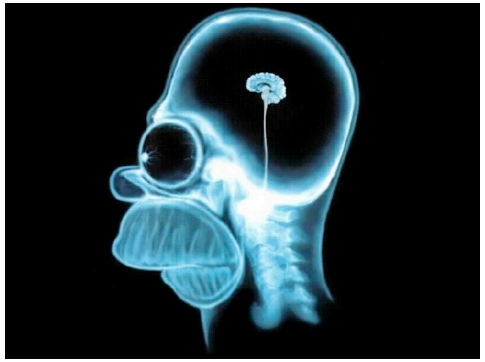
1. Translate center to origin
2. Scale object
3. Rotate object
4. Translate center back



This can get quite messy!

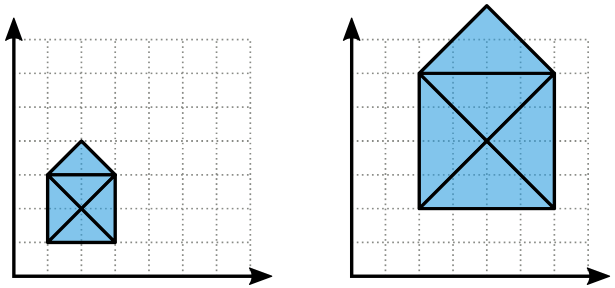
Important Questions

- How to efficiently combine several transformations?

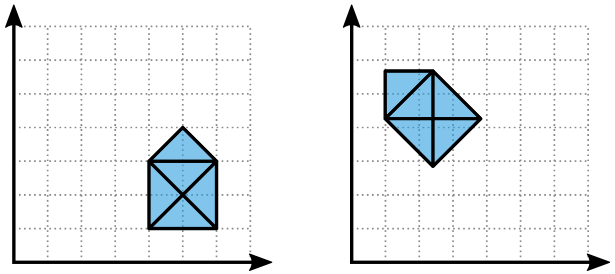


Represent transformations as matrices!

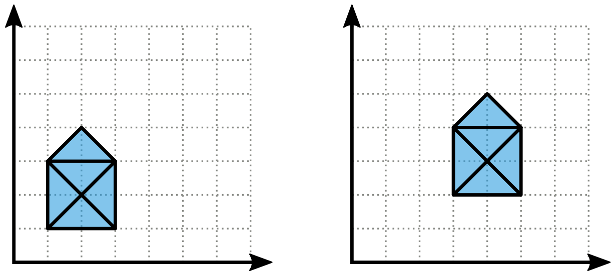
Matrix Representation



scaling



rotation



translation

$$\mathbf{S}\left(s_x, s_y\right) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \qquad \mathbf{R}\left(\theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \mathbf{T}\left(t_x, t_y\right) = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Which transformations can be written as matrices?

Linear Maps & Matrices

- Assume a *linear* transformation $L\colon \mathbb{R}^n \rightarrow \mathbb{R}^n$
    - $L(\mathbf{a} + \mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b})$
    - $L(\alpha\, \mathbf{a}) = \alpha\, L(\mathbf{a})$
- Point $\mathbf{x} = (x_1, \ldots, x_n)^\top$ can be written as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$$

Linear Maps & Matrices

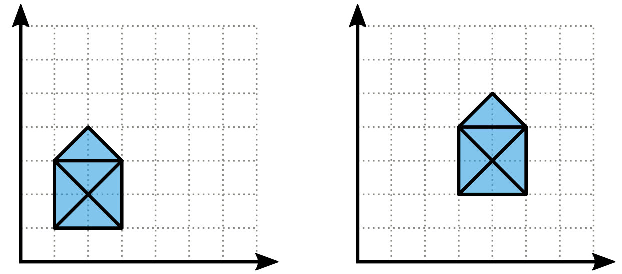
- Exploit linearity of L

$$\begin{aligned} L\left(\mathbf{x}\right) &= L\left(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n\right) \\ &= x_1\, L\left(\mathbf{e}_1\right) + x_2\, L\left(\mathbf{e}_2\right) + \ldots + x_n\, L\left(\mathbf{e}_n\right) \\ &= \underbrace{\left[L\left(\mathbf{e}_1\right),\ L\left(\mathbf{e}_2\right),\ \ldots,\ L\left(\mathbf{e}_n\right)\right]}_{=:\mathbf{L}} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \;=\; \mathbf{L}\,\mathbf{x} \end{aligned}$$

- Every linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$ can be written as a *unique* $(n \times n)$ matrix $\mathbf{L}$ whose columns are the images of the basis vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

VERY useful fact! 👍 VERY-VERY useful! 😍

Linear vs. Affine Transformations



- Every linear transformation has to preserve the origin
    - $L(\mathbf{0}) = \mathbf{L} \cdot \mathbf{0} = \mathbf{0}$
- Translation is not a linear mapping
    - $T(0,0) = (t_x, t_y)$
- Translation is an **affine** transformation
    - affine mapping = linear mapping + translation
    - $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} = \mathbf{L}\mathbf{x} + \mathbf{t}$

But we REALLY want to represent translations as matrices!

Homogeneous Coordinates

- Extend cartesian coordiantes (x, y) to *homogeneous* coordinates (x, y, w)
  - Points are represented by $(x, y, 1)^\top$
  - Vectors are represented by $(x, y, 0)^\top$
- Only homogeneous coordinates with $w \in \{0, 1\}$ are easy to interpret
  - vector + vector = vector
  - point - point = vector
  - point + vector = point
  - point + point = ??
- Only affine combinations of points $\mathbf{x}_i$ make sense
  - $\sum_i \alpha_i \mathbf{x}_i$ with $\sum_i \alpha_i = 1$ (and $\sum_i \alpha_i = 0$)

Homogeneous Coordinates in 2D

- Points
  - Each homogeneous vector of the form $k \cdot (x_1, x_2, x_3)^\top$ with $k, x_3 \neq 0$ represents the same **2D point** $(x_1/x_3, x_2/x_3)^\top$
  - Each homogeneous vector of the form $(x_1, x_2, 0)^\top$ represents the **2D vector** $(x_1, x_2)^\top$
- Lines
  - Each homogeneous vector of the form $k \cdot (a, b, c)^\top$ with $k \neq 0$ represents the same **2D line** $a \cdot x + b \cdot y + c = 0$
- Incidence of points and lines
  - A point with homogenous vector $\mathbf{p} = (x_1, x_2, x_3)^\top$ lies on the line with homogeneous vector $\mathbf{l} = (a, b, c)^\top$ iff $\langle \mathbf{p}, \mathbf{l} \rangle = 0$

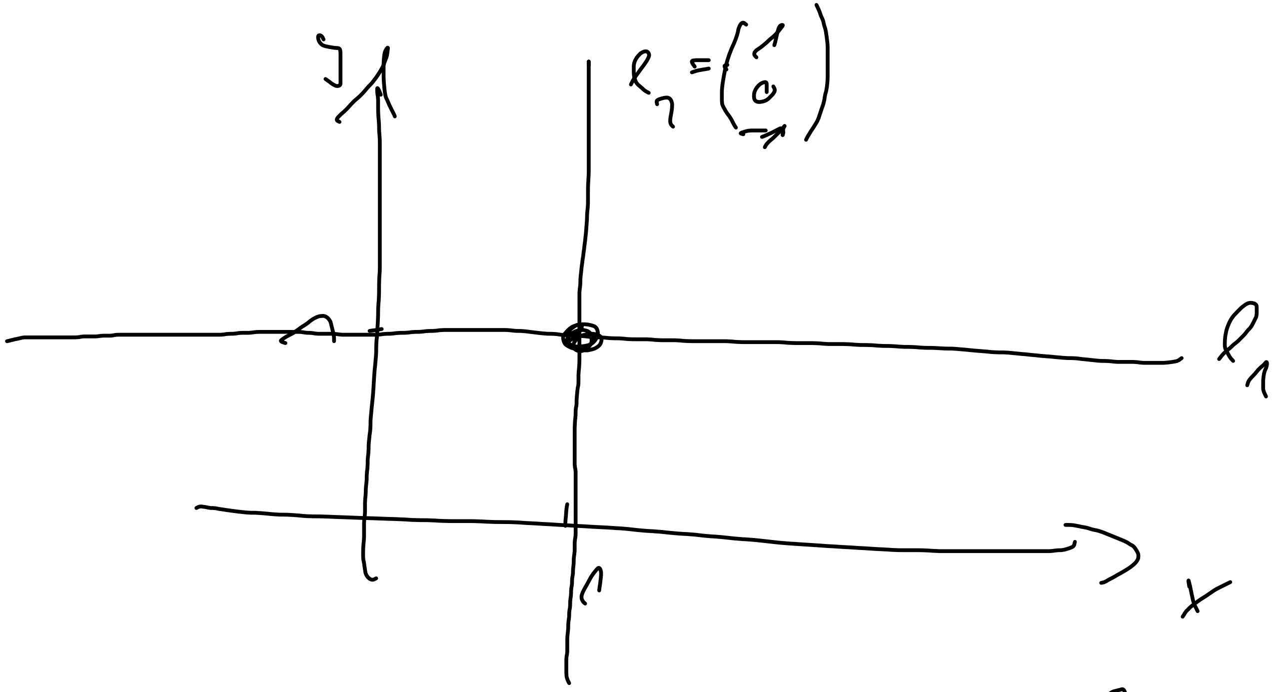
Intersection of two lines in 2D

Two lines with homogenous vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ **intersect** in a point with homogeneous coordinate vector $\mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2 = [\mathbf{l}_1]_\times \mathbf{l}_2$

$x = 1$

$1x + 0 \cdot y - 1 = 0$

$l_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$



$0x + 1 \cdot y - 1 = 0$

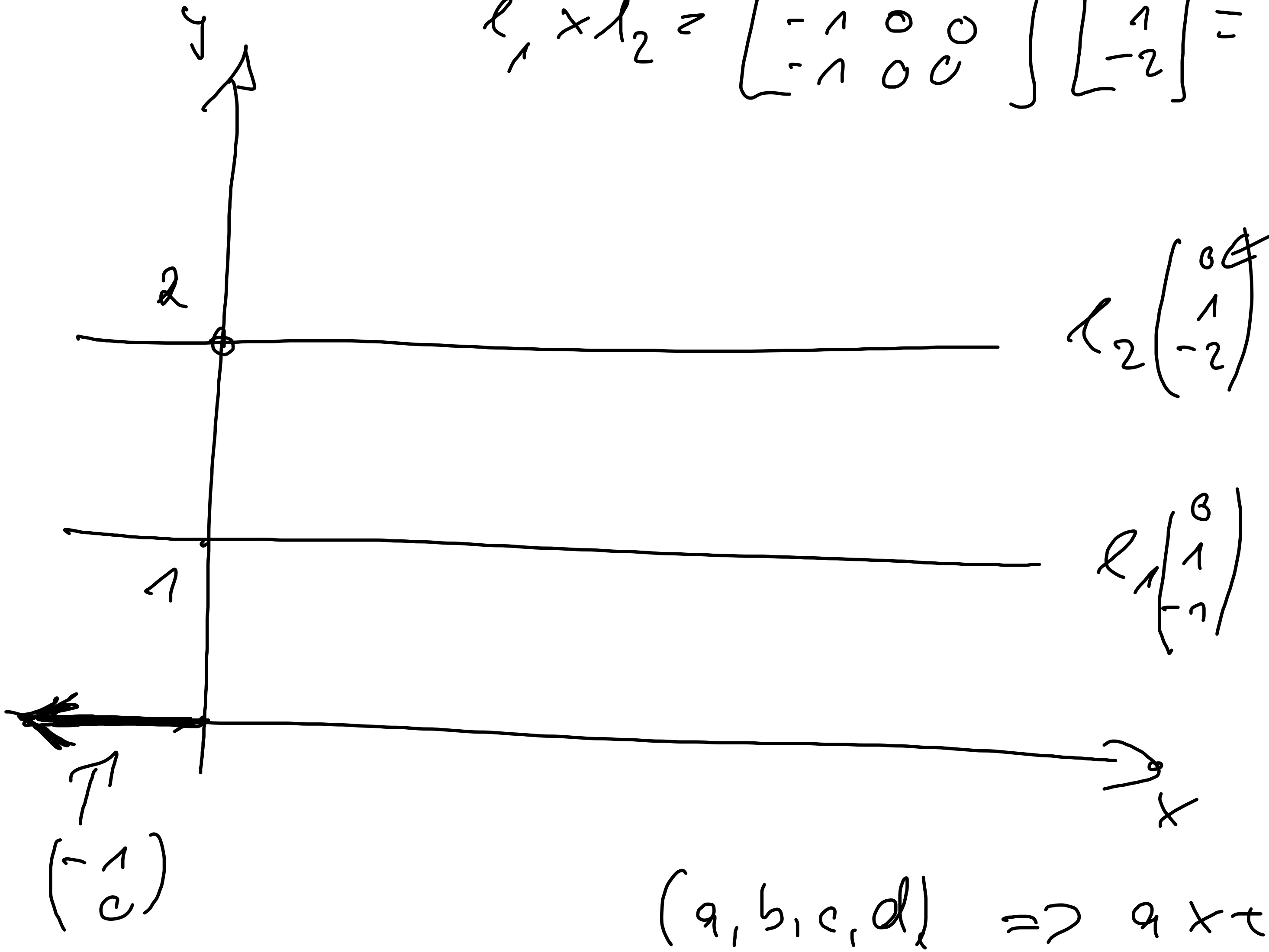
$l_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $y = 1$

$(1, 1, 1)^T$

$\cong$

$$l_1 \times l_2 = [l_1]_\times l_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$l_1 \times l_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$



$\underline{0} \cdot x + \underline{1}\, y - \underline{2} = 0$

$y = 2$

$\triangleq \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$y = 1$

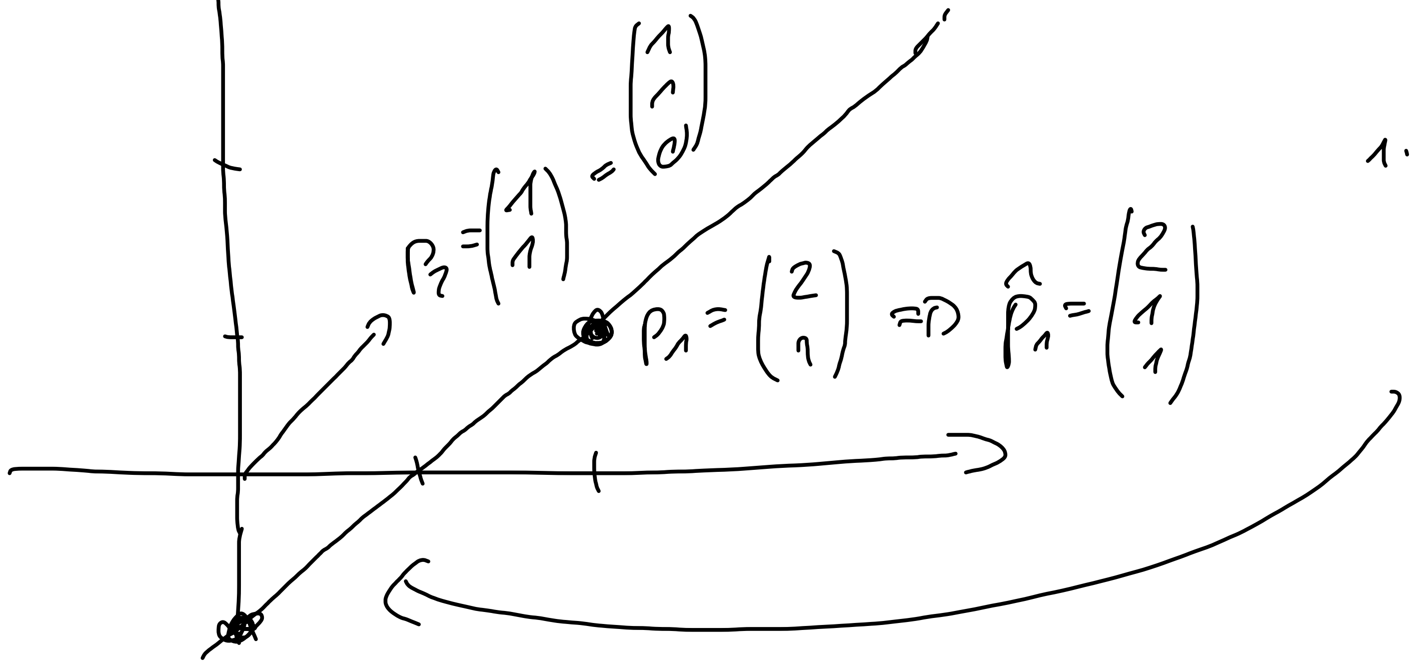
$(a, b, c, d) \Rightarrow a\, x + b \cdot y + c \cdot z + d = 0$

Line given by two points in 2D

The **line given by two points** with homogenous vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ is given by the homogeneous coordinate vector $\mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2 = [\mathbf{p}_1]_\times \mathbf{p}_2$

$$p_2 \times p_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

l



$$p_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$p_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \hat{p}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$1 \cdot x - 1 \cdot y - 1 = 0$$

$$\Updownarrow$$

$$y = x - 1$$

Homogeneous Coordinates

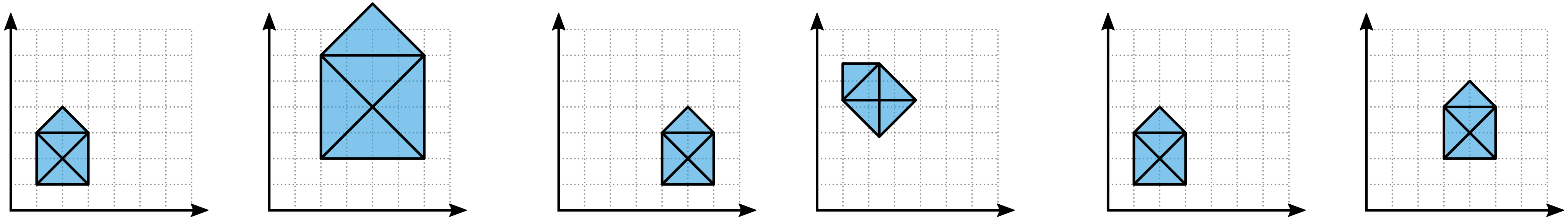
- Matrix representation of translations

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Matrix representation of arbitrary affine transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Matrix Representation



scaling *rotation* *translation*

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Columns of matrix are images of basis vectors!

Concatenation of Transformations

- Apply sequence of affine transformations $\mathbf{A}_1, \ldots, \mathbf{A}_k$
- Concatenate transformations by matrix multiplication

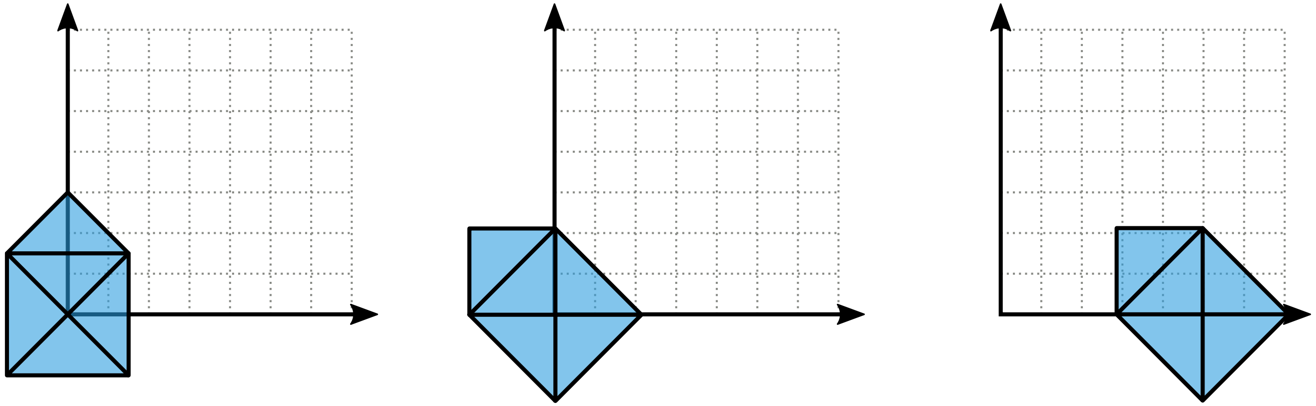
$$A_k\left(\ldots A_2\left(A_1\left(\mathbf{x}\right)\right)\right) \;=\; \underbrace{\mathbf{A}_k \cdots \mathbf{A}_2 \cdot \mathbf{A}_1}_{\mathbf{M}} \cdot \mathbf{x}$$

- Precompute matrix $\mathbf{M}$ and apply it to all (=many!) object vertices.

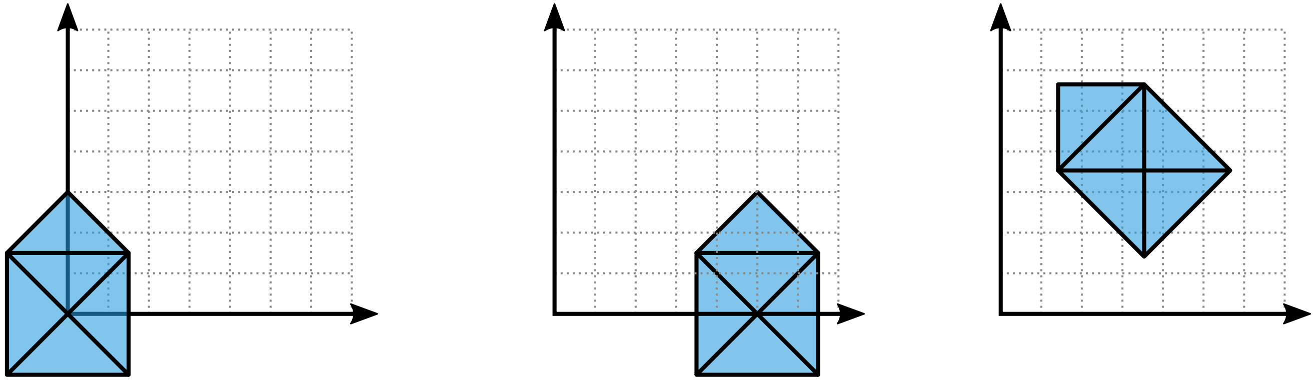
Very important for performance!

Ordering of Matrix Multiplication

- First rotation, then translation



- First translation, then rotation



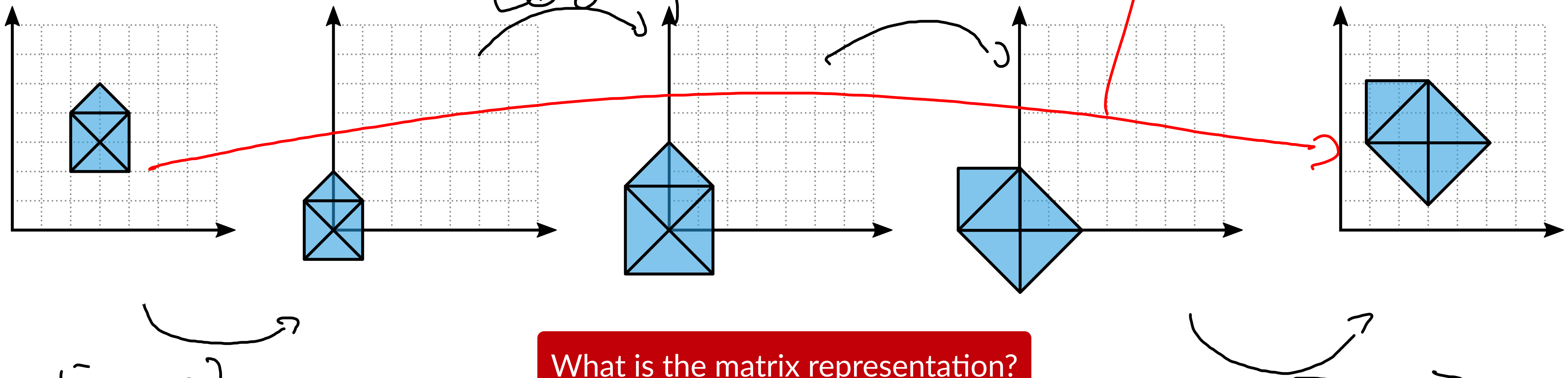
How to rotate/scale around object center?

1. Translate center to origin
2. Scale object
3. Rotate object
4. Translate center back

$$\mathbb{R}^{3\times3} \ni G = \underline{T_2\, R \cdot S \cdot T_1}$$

$$S = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2}/2 & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2}/2 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

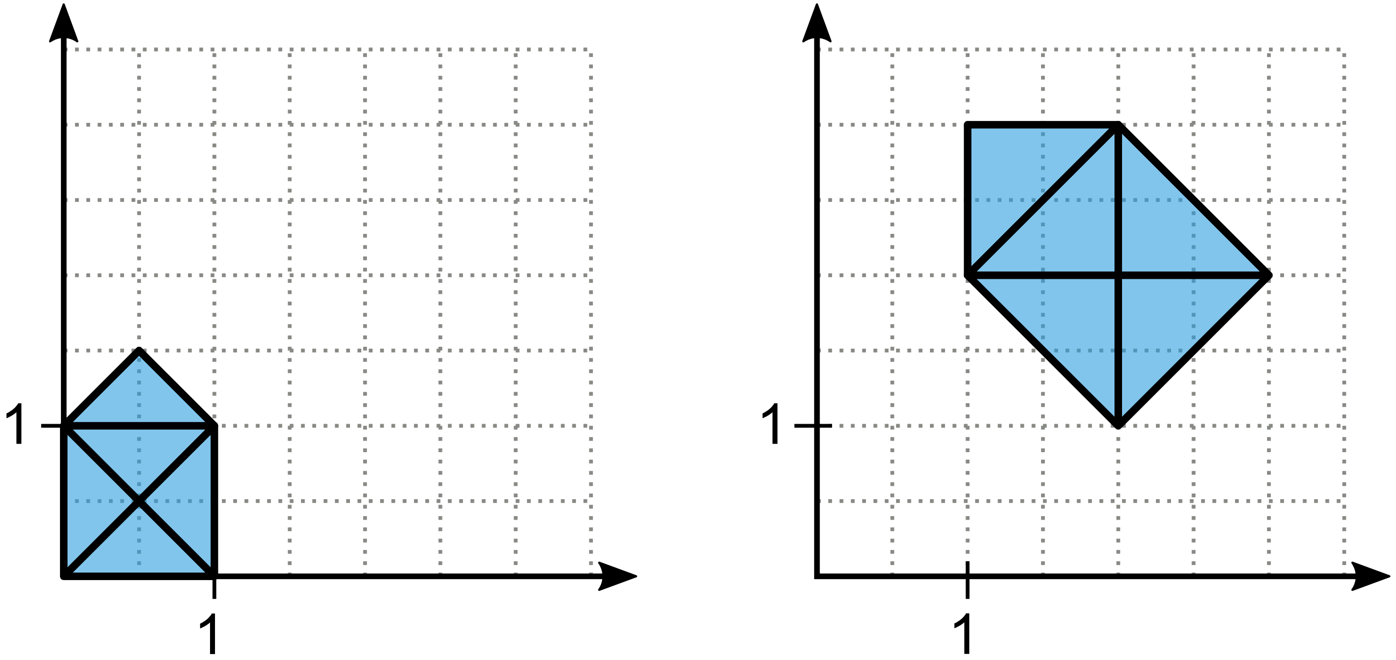


$$T_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the matrix representation?

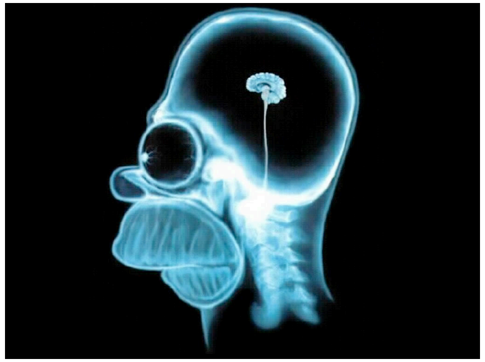
$$T_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Representation?



Important Questions

- What is preserved by affine transformations?
- What is preserved by orthogonal transformations?

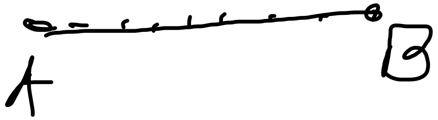


Affine Transformations

- Any point $\mathbf{C}$ on a line is an affine combination

$$(1-\alpha)\mathbf{A} + \alpha\mathbf{B}$$

$\alpha \in [0,1]$

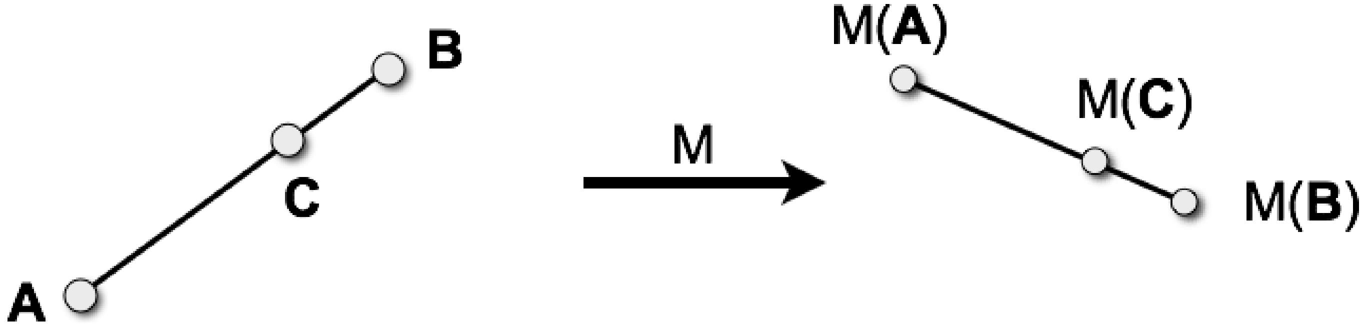


of its endpoints $\mathbf{A}$ and $\mathbf{B}$.

- Affine transformation $\mathbf{M}$ preserves affine combinations

$$\mathbf{M}\left((1-\alpha)\mathbf{A} + \alpha\mathbf{B}\right) \;=\; (1-\alpha)\mathbf{M}\left(\mathbf{A}\right) + \alpha\mathbf{M}\left(\mathbf{B}\right)$$

- Straight lines stay straight lines



Orthogonal Transformations

- A matrix $\mathbf{M}$ is *orthogonal* iff…
    - …its columns are orthonormal vectors
    - …its rows are orthonormal vectors
    - …its inverse is its transposed: $\mathbf{M}^{-1} = \mathbf{M}^\top$
- Orthogonal matrices / mappings…
    - …preserve angles and lengths
    - …can only be rotations or reflections
    - …have determinant +1 or -1

Quiz

Quiz: Transformations

Which matrix represents the 2D translation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + a \\ y + b \end{pmatrix}$?

A: $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

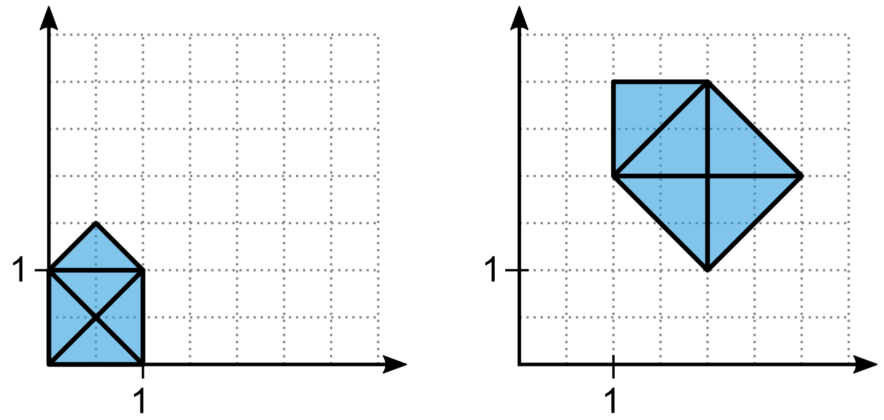
B: $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$

D: $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}$

Quiz: Transformations

Which matrix computes the transformation on the right?



A: $\mathbf{T}(2,1)\cdot\mathbf{S}\cdot\mathbf{R}$

B: $\mathbf{T}(2,2)\cdot\mathbf{S}\cdot\mathbf{R}\cdot\mathbf{T}\left(-\frac{1}{2},-\frac{1}{2}\right)$

C: $\mathbf{R}\cdot\mathbf{S}\cdot\mathbf{T}\left(\frac{3}{2},\frac{3}{2}\right)$

D: $\mathbf{T}\left(-\frac{1}{2},-\frac{1}{2}\right)\cdot\mathbf{R}\cdot\mathbf{S}\cdot\mathbf{T}(2,2)$

Quiz: Lines and Points in 2D

Give the homogenous vector for line through the points $\mathbf{x} = (1, 2)^\top$ and $\mathbf{y} = (3, 4)^\top$.

A: $(1, -1, 1)^\top$

B: $(1, 2, 3)^\top$

C: $(2, 3, 4)^\top$

D: $(-2, 2, -2)^\top$

Quiz: Lines and Points in 2D

Give the homogenous vector for the intersection of the lines $x - y + 1 = 0$ and $x - y - 1 = 0$.

A: $(2, 2, 0)^\top$

B: $(1, 2, 3)^\top$

C: $(1, 1, 0)^\top$

D: $(-2, 2, -2)^\top$