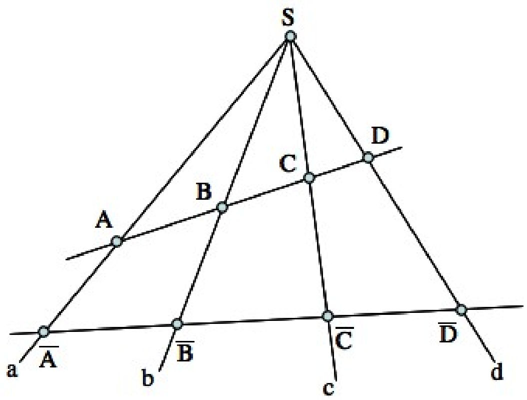
Assignment 2

Welcome to the second assignment of the lecture *3D Vision and Deep Learning* in summer semester 2024. **Please read all instructions carefully!** The goal of this assignment is to get a deeper understanding of projective mappings and the structure of three-dimensional rotations. Submission is due on Monday, April 29th, 2024 at 2pm.

Exercise 1 (2 points)**.** Show that a projective mapping of a straight line g to a straight line $\bar{g}$ preserves the *cross ratio*, i.e. $D(A, B, C, D) = D(\bar{A}, \bar{B}, \bar{C}, \bar{D})$.

Use the fact that the cross ratio can be defined for four points A, B, C, D lying on a straight line as well as for four straight lines a, b, c, d intersecting in a common point as

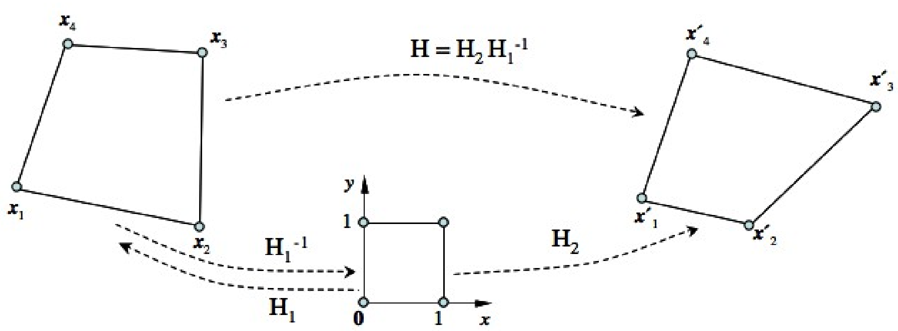
$$D(a, b, c, d) = \frac{\sin(ab)}{\sin(ac)} : \frac{\sin(bd)}{\sin(cd)}$$



Hint: Calculate the aera of triangles using the following two different formulas:

$$F_{\Delta ACS} = \frac{1}{2}|AC|h = \frac{1}{2}|SA||SC|\sin(ac)$$

Exercise 2 (5 points)**.** A *homography* H between two images can be determined using four point correspondences $\mathbf{x}_i, \mathbf{x}'_i$, where $\mathbf{x}_i = (x_i, y_i)$ is a point in the source and $\mathbf{x}'_i = (x'_i, y'_i)$ its corresponding point in the target image. An alternative to directly solving the emerging linear system for the 8 unknowns of H is a two-stage mapping over a unit square using the identity $H = H_2 H_1^{-1}$ (see the figure below).



For the homography $H = (h_{ij})_{\substack{i=1,2,3\\ j=1,2,3}}$ that maps the unit square according to

$$(0, 0) \to \mathbf{x}_1, \quad (1, 0) \to \mathbf{x}_2, \quad (1, 1) \to \mathbf{x}_3, \quad (0, 1) \to \mathbf{x}_4$$

onto the four points $\mathbf{x}_i = (x_i, y_i)$, $i = 1, \ldots, 4$ it holds

$$\begin{aligned} h_{11} &= x_2 - x_1 + h_{31}x_2, & h_{12} &= x_4 - x_1 + h_{32}x_4, & h_{13} &= x_1 \\ h_{21} &= y_2 - y_1 + h_{31}y_2, & h_{22} &= y_4 - y_1 + h_{32}y_4, & h_{23} &= y_1 \end{aligned}$$

with

$$\begin{aligned} h_{31} &= \frac{(x_1 - x_2 + x_3 - x_4)(y_4 - y_3) - (y_1 - y_2 + y_3 - y_4)(x_4 - x_3)}{(x_2 - x_3)(y_4 - y_3) - (x_4 - x_3)(y_2 - y_3)} \\ h_{32} &= \frac{(y_1 - y_2 + y_3 - y_4)(x_2 - x_3) - (x_1 - x_2 + x_3 - x_4)(y_2 - y_3)}{(x_2 - x_3)(y_4 - y_3) - (x_4 - x_3)(y_2 - y_3)} \end{aligned}$$

1. (1 point) Determine the homography H, that maps the four points

   $$\mathbf{x}_1 = (2, 5), \quad \mathbf{x}_2 = (4, 6), \quad \mathbf{x}_3 = (7, 9), \quad \mathbf{x}_4 = (5, 9)$$

   to the four corresponding points

   $$\mathbf{x}'_1 = (4, 3), \quad \mathbf{x}'_2 = (5, 2), \quad \mathbf{x}'_3 = (9, 3), \quad \mathbf{x}'_4 = (7, 5)$$

   using the described two-stage mapping over the unit square. Thereby, first determine the matrices H_1, H_2 und H_1^{-1}.

2. (4 points) Write a `python` script using *OpenCV* that allows to select four points in an image using the mouse. After the four points are selected the complete image should be transformed by a homography that is determined by the four selected points and the four original corners of the image as corresponding points. Finally your program should display the transformed image.

Exercise 3 (5 points)**.** A map $g : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is called a *rigid-body motion* or a *special Euclidean transformation* if it preserves the norm and the cross product of any two vectors, i.e.

1. *norm*: $\|g(\mathbf{u})\| = \|\mathbf{u}\| \;\; \forall \;\; \mathbf{u} \in \mathbb{R}^3$.
2. *cross product*: $g(\mathbf{u} \times \mathbf{v}) = g(\mathbf{u}) \times g(\mathbf{v}) \;\; \forall \;\; \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

The collection of all such transfomations (motions) is denoted by $SE(3)$. Given a rotation matrix $R \in SO(3)$, its action on a vector $\mathbf{u}$ is defined as $R\mathbf{u}$. Prove that any rotation matrix preserve both the inner product and the cross product of two vectors. Therefore, a rotation is indeed a rigid-body motion.

Hint: You can use the fact, that for any regular matrix $A \in \mathbb{R}^{3\times3}$ and any vector $\mathbf{v} \in \mathbb{R}^3$ it holds $[A\mathbf{v}]_\times = \det(A)A^{-\top}[\mathbf{v}]_\times A^{-1}$.