3D Computer Vision (SoSe2024)

Primitives and Transformations in 3D

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Last lecture

Last lecture

- 1D Transformations
- Primitives and Transformations
  - Homogeneous Coordinates
  - Points, Lines and Planes
  - 2D Transformations
  - Homography Estimation

Today's Lecture

Today's Lecture

- The 3D Projective Space $\mathbb{P}^3$
  - Points, Planes,
  - Straight Lines, …
  - … and their transformations

Reminder

Points

Points in 1D/2D/3D can be written in **inhomogeneous coordinates** as

$$x \in \mathbb{R} \quad \text{or} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \quad \text{or} \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

or in **homogenous coordinates** as

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} \in \mathbb{P}^1 \quad \text{or} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} \in \mathbb{P}^2 \quad \text{or} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} \in \mathbb{P}^3$$

where $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ is called **projective space**. Homogeneous vectors that differ only by scale are considered equivalent and define an equivalence class, thus homogeneous vectors are **defined only up to scale**.

Points

An **inhomogeneous vector** $\mathbf{x}$ is converted to a **homogeneous vector** $\tilde{\mathbf{x}}$ as follows

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix} = \bar{x} \quad \text{or} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \bar{\mathbf{x}} \quad \text{or} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \bar{\mathbf{x}}$$

with the **augmented vector** $\bar{\mathbf{x}}$. To convert in opposite direction one has to divide by $\tilde{w}$:

$$\bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \frac{1}{\tilde{w}} \tilde{\mathbf{x}} = \frac{1}{\tilde{w}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ \tilde{z}/\tilde{w} \\ 1 \end{pmatrix}$$

Homogeneous points whose last element is $\tilde{w} = 0$ can't be represented with inhomogeneous coordinates. They are called **ideal points** or **points at infinity**.

2D Lines

2D lines can also be expressed using homogeneous coordinates $\tilde{\mathbf{l}} = (a, b, c)^\top$

$$\{\tilde{\mathbf{x}} \mid \tilde{\mathbf{l}}\tilde{\mathbf{x}} = 0\} \quad \Leftrightarrow \quad \{x, y \mid ax + by + c = 0\}$$

- We can normalize $\tilde{\mathbf{l}}$ to $\tilde{\mathbf{l}} = (n_x, n_y, d)^\top = (\mathbf{n}, d)^\top$ with $\|\mathbf{n}\|_2 = 1$ (**Hesse normal form**)
  - In this case, $\mathbf{n}$ is the **normal vector** perpendicular to the line and d is its **distance** to the origin.
- An exception is the **line at infinity** $\tilde{\mathbf{l}}_\infty = (0, 0, 1)^\top$ which passes through all ideal points.

Duality (in 2D)

The **duality principle**:

For every proposition of two-dimensional projective geometry there exists a ***dual proposition*** *which is obtained by interchanging the role of points and lines in the original proposition.*

- For example, dual are

$$\tilde{\mathbf{x}} \quad \longleftrightarrow \quad \tilde{\mathbf{l}}$$

$$\tilde{\mathbf{x}}^T\tilde{\mathbf{l}} = 0 \quad \longleftrightarrow \quad \tilde{\mathbf{l}}^T\tilde{\mathbf{x}} = 0$$

$$\tilde{\mathbf{x}} = \tilde{\mathbf{l}} \times \tilde{\mathbf{l}}' \quad \longleftrightarrow \quad \tilde{\mathbf{l}} = \tilde{\mathbf{x}} \times \tilde{\mathbf{x}}'$$

Overview of 2D Transformations

Transformation	Matrix	# DOF	Preserves
translation	$\begin{bmatrix}\mathbf{I} & \mathbf{t}\end{bmatrix}_{2\times 3}$	2	orientation
rigid (Euclidean)	$\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}_{2\times 3}$	3	length
similarity	$\begin{bmatrix}s\mathbf{R} & \mathbf{t}\end{bmatrix}_{2\times 3}$	4	angles
affine	$\begin{bmatrix}\mathbf{A} & \mathbf{t}\end{bmatrix}_{2\times 3}$	6	parallelism
projective	$\begin{bmatrix}\mathbf{H}\end{bmatrix}_{3\times 3}$	8	straight lines

- Transformations form **nested set of groups** (closed under composition, inverse)
- 2×3 matrices are extended with a third $[\mathbf{0}^\top 1]$ row for homogeneous transforms

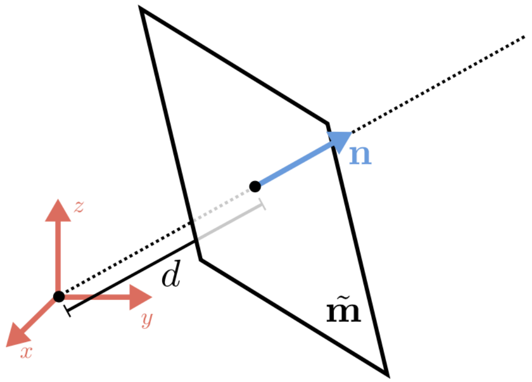
The Projective Space $\mathbb{P}^3$

3D Planes

3D planes can also be represented with homogeneous coordinates $\tilde{\mathbf{m}} = (a, b, c, d)^\top$ as

$$\{\tilde{\mathbf{x}} \mid \tilde{\mathbf{m}}^\top \bar{\mathbf{x}} = 0\} \quad \Leftrightarrow \quad \{x, y, z \mid ax + by + cz + d = 0\}$$

- $\tilde{\mathbf{m}}$ can be **normalized:** $\tilde{\mathbf{m}} = (n_x, n_y, n_z, d)^\top = (\mathbf{n}, d)^\top$, $\|\mathbf{n}\|_2 = 1$ (**Hesse normal form**)
    - $\mathbf{n}$ is the **normal vector** the plane and d its **distance** to the origin



$$\underbrace{\begin{pmatrix} n_x & n_y & n_z & d \end{pmatrix}}_{\tilde{\mathbf{m}}^T} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

- An exception is the **plane at infinity** $\tilde{\mathbf{m}} = (0, 0, 0, 1)^\top$ which passes through all ideal points (=points at infinity) for which $\tilde{w} = 0$

Plane given by Three Points in $\mathbb{P}^3$

- For a plane $\tilde{\mathbf{m}}$ that contains the three points $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3$ (in general position) it must hold $\tilde{\mathbf{x}}_1^\top \tilde{\mathbf{m}} = \tilde{\mathbf{x}}_2^\top \tilde{\mathbf{m}} = \tilde{\mathbf{x}}_3^\top \tilde{\mathbf{m}} = 0$ or

$$\begin{bmatrix} \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \\ \tilde{\mathbf{x}}_3^\top \end{bmatrix} \tilde{\mathbf{m}} = 0$$

- Thus the plane $\tilde{\mathbf{m}}$ that contains the three points $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3$ is the **right null space** of

$$\begin{bmatrix} \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \\ \tilde{\mathbf{x}}_3^\top \end{bmatrix} = \begin{pmatrix} x_{1_1} & x_{1_2} & x_{1_3} & x_{1_4} \\ x_{2_1} & x_{2_2} & x_{2_3} & x_{2_4} \\ x_{3_1} & x_{3_2} & x_{3_3} & x_{3_4} \end{pmatrix}$$

- Does that relate to calculating the straight line through two points?

Reminder: Straight Line given by Two Points in $\mathbb{P}^2$

- A line $\tilde{\mathbf{l}}$ in $\mathbb{P}^2$ through $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2$ (in general position) is given by the **right null space** of

$$\begin{bmatrix} \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \end{bmatrix} = \begin{pmatrix} x_{1_1} & x_{1_2} & x_{1_3} \\ x_{2_1} & x_{2_2} & x_{2_3} \end{pmatrix}$$

  which can be calculated directly by the cross product, i.e. $\tilde{\mathbf{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$

- As any point $\tilde{\mathbf{x}}$ on the line $\tilde{\mathbf{l}}$ is a linear combination of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2$ it must hold

$$\det \begin{bmatrix} \tilde{\mathbf{x}}^\top \\ \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \end{bmatrix} = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_{1_1} & x_{1_2} & x_{1_3} \\ x_{2_1} & x_{2_2} & x_{2_3} \end{pmatrix} = 0$$

- Laplace expansion results in $0 = x_{_1} D_{23} - x_{_2} D_{13} + x_{_3} D_{12}$ with $D_{ij} = \det \begin{pmatrix} x_{1_i} & x_{1_j} \\ x_{2_i} & x_{2_j} \end{pmatrix}$, i.e. $\tilde{\mathbf{l}} = (D_{23}, -D_{13}, D_{12})^\top$

Plane given by Three Points in $\mathbb{P}^3$

- As any point $\tilde{\mathbf{x}}$ on the plane $\tilde{\mathbf{m}}$ is a linear combination of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3$, it must hold

$$\det \begin{bmatrix} \tilde{\mathbf{x}}^\top \\ \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \\ \tilde{\mathbf{x}}_3^\top \end{bmatrix} = \det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_{1_1} & x_{1_2} & x_{1_3} & x_{1_4} \\ x_{2_1} & x_{2_2} & x_{2_3} & x_{2_4} \\ x_{3_1} & x_{3_2} & x_{3_3} & x_{3_4} \end{pmatrix} = 0$$

- Laplace expansion results in

$$0 = x_1 D_{234} - x_2 D_{134} + x_3 D_{124} + x_4 D_{123} \quad \text{with} \quad D_{ijk} = \det \begin{pmatrix} x_{1_i} & x_{1_j} & x_{1_k} \\ x_{2_i} & x_{2_j} & x_{2_k} \\ x_{3_i} & x_{3_j} & x_{3_k} \end{pmatrix}$$

and thus $\tilde{\mathbf{m}} = (D_{234}, -D_{134}, D_{124}, -D_{123})^\top$

Example: Plane given by Three non-vanishing Points in $\mathbb{P}^3$

- For the plane $\tilde{\mathbf{m}}$ through the three non-vanishing points

$$\tilde{\mathbf{x}}_1 = \begin{pmatrix} \mathbf{x}_1 \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{x}}_2 = \begin{pmatrix} \mathbf{x}_2 \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{x}}_3 = \begin{pmatrix} \mathbf{x}_3 \\ 1 \end{pmatrix}$$

it holds e.g.

$$D_{234} = \det \begin{pmatrix} x_{1_2} & x_{1_3} & 1 \\ x_{2_2} & x_{2_3} & 1 \\ x_{3_2} & x_{3_3} & 1 \end{pmatrix} = \det \begin{pmatrix} x_{1_2} - x_{3_2} & x_{1_3} - x_{3_3} & 0 \\ x_{2_2} - x_{3_2} & x_{2_3} - x_{3_3} & 0 \\ x_{3_2} & x_{3_3} & 1 \end{pmatrix} = ((\mathbf{x}_1 - \mathbf{x}_3) \times (\mathbf{x}_2 - \mathbf{x}_3))_1$$

and thus

$$\tilde{\mathbf{m}} = \begin{bmatrix} (\mathbf{x}_1 - \mathbf{x}_3) \times (\mathbf{x}_2 - \mathbf{x}_3) \\ -\mathbf{x}_3^\top (\mathbf{x}_1 \times \mathbf{x}_2) \end{bmatrix}$$

Duality (in 3D)

The **duality principle**:

*For every proposition of three-dimensional projective geometry there exists a **dual proposition** which is obtained by interchanging the role of points and planes in the original proposition.*

- Dual are

$$\tilde{\mathbf{x}} \quad \longleftrightarrow \quad \tilde{\mathbf{m}}$$

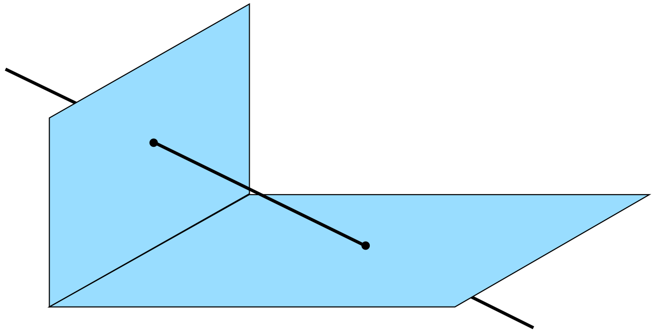
$$\tilde{\mathbf{x}}^T \tilde{\mathbf{m}} = 0 \quad \longleftrightarrow \quad \tilde{\mathbf{m}}^T \tilde{\mathbf{x}} = 0$$

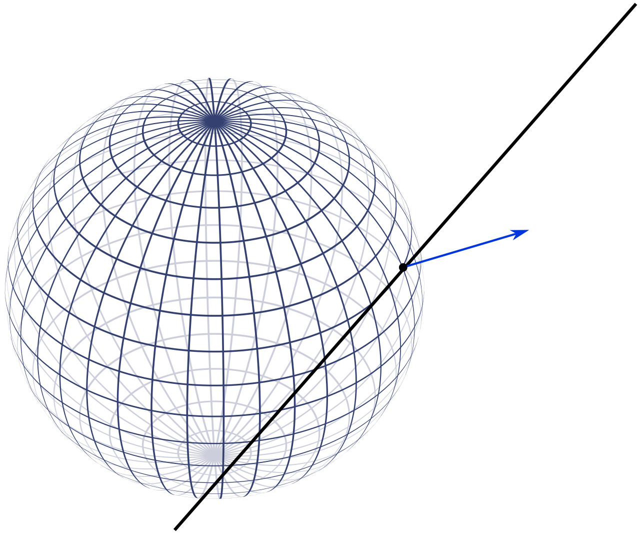
$\tilde{\mathbf{x}}$ is right null space of $[\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2, \tilde{\mathbf{m}}_3]^\top$ $\longleftrightarrow$ $\tilde{\mathbf{m}}$ is right null space of $[\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3]^\top$

Straight Lines are dual to Straight Lines

Straight Lines in Projective Space $\mathbb{P}^3$

- Representing a straight line in 3D is problematic
  - Straight lines in 3D have 4 DOF
  - Thus they would have to be represented by a homogeneous vector with five elements
  - Are there more suitable representations?
- Usual representations for straight lines in space are
  - Linear combination of two points
  - Plücker matrices
  - Plücker coordinates



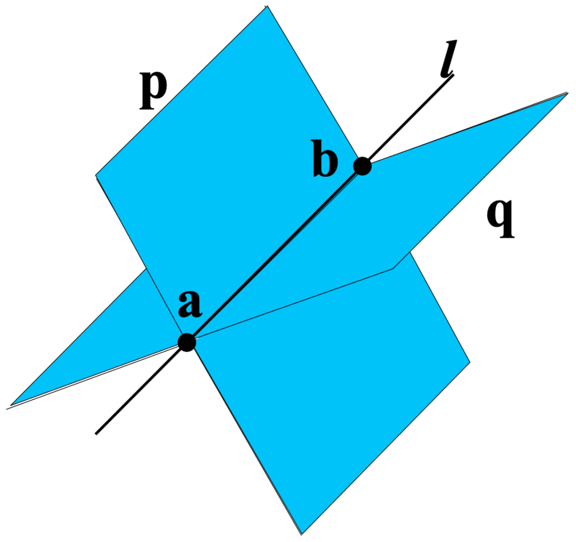


Straight Lines in $\mathbb{P}^3$ as Linear Combination of Points

- A straight line $\mathbf{l}$ can be represented as the space spanned by the matrix $\tilde{\mathbf{W}}$, consisting of points (on a straight line) $\mathbf{a}$ and $\mathbf{b}$

$$\tilde{\mathbf{W}} = \begin{bmatrix} \tilde{\mathbf{a}}^\top \\ \tilde{\mathbf{b}}^\top \end{bmatrix}$$

  - The **span** of $\tilde{\mathbf{W}}^\top$ ist the bundle of points $\lambda\tilde{\mathbf{a}} + \mu\tilde{\mathbf{b}}$ on the straight line $\mathbf{l}$
  - The **span of the 2-dimensional right null space of** $\tilde{\mathbf{W}}$ is the bundle of planes with the straight line $\mathbf{l}$ as axis



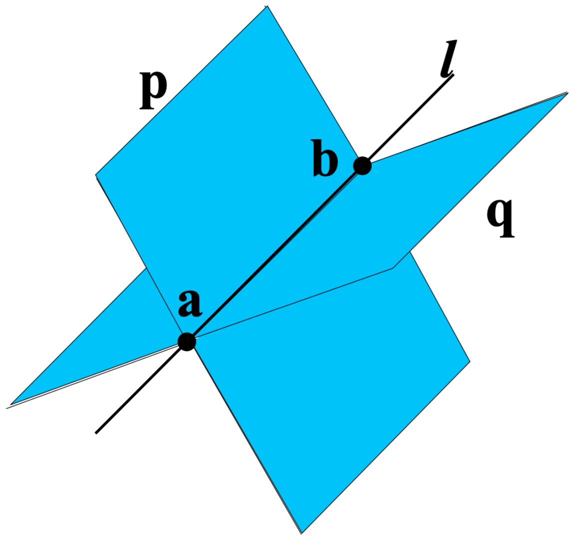
Straight Lines in $\mathbb{P}^3$ as Linear Combination of Planes

- The **dual representation** as the right null space of the matrix

$$\tilde{\mathbf{W}}^* = \begin{bmatrix} \tilde{\mathbf{p}}^\top \\ \tilde{\mathbf{q}}^\top \end{bmatrix}$$

contains the planes $\mathbf{p}$ and $\mathbf{q}$, that intersect in the straight line $\mathbf{l}$

  - The **span** of $\tilde{\mathbf{W}}^{*\top}$ ist the bundle of planes $\lambda'\tilde{\mathbf{p}} + \mu'\tilde{\mathbf{q}}$ with the axis $\mathbf{l}$
  - The **span of the 2-dimensional right null space of** $\tilde{\mathbf{W}}^*$ is the bundle of points on the straight line $\mathbf{l}$ as axis



- Relationship between the two representations:

$$\tilde{\mathbf{W}}^*\tilde{\mathbf{W}}^\top = \tilde{\mathbf{W}}\tilde{\mathbf{W}}^{*\top} = \mathbf{0}_{2\times 2}$$

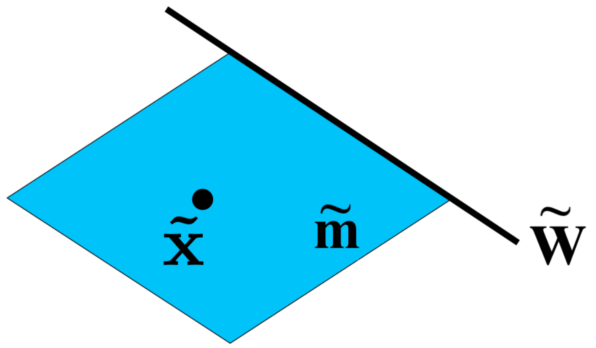
Example

- The x**-axis** ...
  - ... is spannend by the point $\mathbf{a} = (0, 0, 0)^\top$ and the direction of the x-axis $\mathbf{b} = (1, 0, 0)^\top$ and thus
  $$\tilde{\mathbf{W}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
  - ... is the right null space of the xy-plane $\tilde{\mathbf{p}} = (0, 0, 1, 0)^\top$ and the xz-plane $\tilde{\mathbf{p}} = (0, 1, 0, 0)^\top$ and thus
  $$\tilde{\mathbf{W}}^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Union and Intersection of Points, Lines and Planes

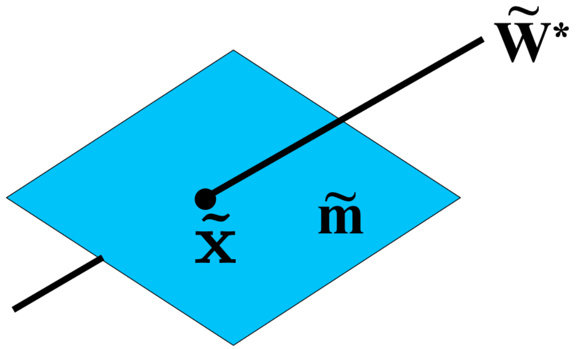
- The plane $\tilde{\mathbf{m}}$, given by the union of the straight line $\tilde{\mathbf{W}}$ with the point $\tilde{\mathbf{x}}$ is the right null space of the matrix $\tilde{\mathbf{M}}$, containing $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{x}}$, i.e.

$$\tilde{\mathbf{M}}\tilde{\mathbf{m}} = \mathbf{0} \quad \text{with} \quad \tilde{\mathbf{M}} = \begin{bmatrix} \tilde{\mathbf{W}} \\ \tilde{\mathbf{x}}^\top \end{bmatrix}$$



- The point $\tilde{\mathbf{x}}$, given by the intersection of the plane $\tilde{\mathbf{m}}$ and the straight line $\tilde{\mathbf{W}}^*$ is the right null space of the matrix $\tilde{\mathbf{M}}$, containing $\tilde{\mathbf{W}}^*$ and $\tilde{\mathbf{m}}$, i.e.

$$\tilde{\mathbf{M}}\tilde{\mathbf{x}} = \mathbf{0} \quad \text{with} \quad \tilde{\mathbf{M}} = \begin{bmatrix} \tilde{\mathbf{W}}^* \\ \tilde{\mathbf{m}}^\top \end{bmatrix}$$



Example

- The plane $\tilde{\mathbf{m}}$, given by the union of the straight line $\tilde{\mathbf{W}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and the point $(0,0,1)^\top$ is the right null space of

$$\tilde{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{which results in } \tilde{\mathbf{m}} = (0,1,0,0)^\top$$

- The point $\tilde{\mathbf{p}}$, given by the intersection of the plane $z = 1$ and the straight line $\tilde{\mathbf{W}}^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is the right null space of

$$\tilde{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{which results in } \tilde{\mathbf{p}} = (1, 0, 0, 0)^\top$$

Plücker Coordinates

- For the line given by the two points $\mathbf{a}, \mathbf{b}$ it holds $\tilde{\mathbf{W}} = \begin{bmatrix} \tilde{\mathbf{a}}^\top \\ \tilde{\mathbf{b}}^\top \end{bmatrix}, \;\text{rank}(\tilde{\mathbf{W}}) = 2$

- For two other points $\mathbf{a}', \mathbf{b}'$ on the same line $\mathbf{l}$ it holds

$$\tilde{\mathbf{W}}' = \begin{bmatrix} \tilde{\mathbf{a}'}^\top \\ \tilde{\mathbf{b}'}^\top \end{bmatrix}, \;\text{rank}(\tilde{\mathbf{W}}') = 2 \;\text{ and }\; \tilde{\mathbf{W}}' = \Lambda \tilde{\mathbf{W}} \;\text{ with } \Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \;\det(\Lambda) \neq 0$$

- In particular we get

$$\begin{pmatrix} a'_i & b'_i \\ a'_j & b'_j \end{pmatrix} = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \implies \det \begin{pmatrix} a'_i & b'_i \\ a'_j & b'_j \end{pmatrix} = \det\Lambda \cdot \underbrace{\det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}}_{=:l_{ij}}, \quad i, j = 1, \ldots, 4$$

- As $l_{ii} = 0$, $\binom{4}{2} = 6$ independent quantities remain: $(l_{12} : l_{13} : l_{14} : l_{23} : l_{24} : l_{34})$

Plücker Matrix

- The **Plücker Matrix** $\tilde{\mathbf{L}}$ describes the straight line through the two points $\mathbf{a}$, $\mathbf{b}$:

$$\tilde{\mathbf{L}} = \tilde{\mathbf{a}}\tilde{\mathbf{b}}^\top - \tilde{\mathbf{b}}\tilde{\mathbf{a}}^\top = \begin{pmatrix} 0 & l_{12} & l_{13} & l_{14} \\ -l_{12} & 0 & l_{23} & l_{24} \\ -l_{13} & -l_{23} & 0 & l_{34} \\ -l_{14} & -l_{24} & -l_{34} & 0 \end{pmatrix}, \quad i.e. \;\; l_{ij} = a_i b_j - b_i a_j$$

- Example: Representing the x-axis as Plücker matrix, defined by the two points $\tilde{\mathbf{a}} = (0,0,0,1)^\top$ and $\tilde{\mathbf{b}} = (1,0,0,0)^\top$:

$$\tilde{\mathbf{L}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (1, 0, 0, 0) - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (0, 0, 0, 1) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Properties of Plücker Matrices

- $\tilde{\mathbf{L}}$ is a **skew symmetric homogeneous** 4×4 Matrix
- rank$(\mathbf{L}) = 2$
    - 2-dim null space is the bundle of planes defined by the straight line (it holds $\tilde{\mathbf{L}}\tilde{\mathbf{W}}^{*\top} = \mathbf{0}_{4\times 2}$)
- 4 DOF
    - 6 independet entries - homogenity - det($\tilde{\mathbf{L}} = 0$)
- $\tilde{\mathbf{L}} = \tilde{\mathbf{a}}\tilde{\mathbf{b}}^\top - \tilde{\mathbf{b}}\tilde{\mathbf{a}}^\top$
    - gereralization of the crossproduct to 4D space (6 subdeterminants of $[\tilde{\mathbf{a}}, \tilde{\mathbf{b}}]$)
- $\tilde{\mathbf{L}}$ is independent of the choice of $\mathbf{a}, \mathbf{b}$ on the line, as for any $\tilde{\mathbf{c}} = \tilde{\mathbf{a}} + \mu\tilde{\mathbf{b}}$ it holds

$$\tilde{\mathbf{L}}' = \tilde{\mathbf{a}}\tilde{\mathbf{c}}^\top - \tilde{\mathbf{c}}\tilde{\mathbf{a}}^\top = \tilde{\mathbf{a}}(\tilde{\mathbf{a}}^\top + \mu\tilde{\mathbf{b}}^\top) - (\tilde{\mathbf{a}} + \mu\tilde{\mathbf{b}})\tilde{\mathbf{a}}^\top = \tilde{\mathbf{a}}\tilde{\mathbf{b}}^\top - \tilde{\mathbf{b}}\tilde{\mathbf{a}}^\top = \tilde{\mathbf{L}}$$

- If $\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$ is a point transform, the Plücker Matrix is transformed by $\tilde{\mathbf{L}}' = \tilde{\mathbf{H}}\tilde{\mathbf{L}}\tilde{\mathbf{H}}^\top$

Dual Plücker Matrices

- The dual Plücker matrix $\tilde{\mathbf{L}}^* = \tilde{\mathbf{p}}\tilde{\mathbf{q}}^\top - \tilde{\mathbf{q}}\tilde{\mathbf{p}}^\top$ describes the intersection of planes $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$
- Transformation ot the dual Plücker matrix $\tilde{\mathbf{L}}^*$ with the homography $\tilde{\mathbf{H}}$ is given by $(\tilde{\mathbf{H}}^{-1})^\top \tilde{\mathbf{L}}^* \tilde{\mathbf{H}}^{-1}$
- Relationship between the Plücker matrix $\tilde{\mathbf{L}}$ and its dual $\tilde{\mathbf{L}}^*$:

$$l_{12} : l_{13} : l_{13} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}$$

- Union and Incidence
  - Plane through point and straight line: $\tilde{\mathbf{m}} = \tilde{\mathbf{L}}^* \tilde{\mathbf{x}}$
  - Point on straight line: $\tilde{\mathbf{L}}^* \tilde{\mathbf{x}} = \mathbf{0}$
  - Intersection between plane and straight line: $\tilde{\mathbf{x}} = \tilde{\mathbf{L}} \tilde{\mathbf{m}}$
  - Straight line in plane: $\tilde{\mathbf{L}}^* \tilde{\mathbf{m}} = \mathbf{0}$
  - Coplanar straight lines: $[\tilde{\mathbf{L}}_1, \tilde{\mathbf{L}}_2, \ldots] \tilde{\mathbf{x}} = \mathbf{0}$

Example: Intersection and the Plücker Matrix

- For the intersection $\tilde{\mathbf{x}}$ of the x-axis with the plane $x = 1$ it holds $\tilde{\mathbf{x}} = \tilde{\mathbf{L}}\tilde{\mathbf{m}}$ and thus

$$\tilde{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Plücker Coordinates for non-vanishing Points

- For non-vanishing points $\mathbf{a}, \mathbf{b}$, i.e. $\tilde{\mathbf{a}} = (a_1, a_2, a_3, 1)^\top, \tilde{\mathbf{b}} = (b_1, b_2, b_3, 1)^\top$ it holds

$$(l_{41}, l_{42}, l_{43})^\top = \mathbf{b} - \mathbf{a} \quad \text{and} \quad (l_{23}, l_{31}, l_{12})^\top = \mathbf{a} \times \mathbf{b}$$

- The l_{ij} can be interpreted as a homogeneous vector

$$\tilde{\mathbf{l}} = (l_{41}, l_{42}, l_{43}, l_{23}, l_{31}, l_{12})^\top \in \mathbb{P}^5$$

  - They are called **Plücker coordinates** if the condition $\det(L) = 0$ holds, i.e. if

$$(l_{41}, l_{42}, l_{43})(l_{23}, l_{31}, l_{12})^\top = l_{41}l_{23} + l_{42}l_{31} + l_{43}l_{12} = 0$$

    (only then the vector represents a straight line)

Properties for Plücker Coordinates

- For the two straight lines $\tilde{\mathbf{l}}^1 = (\underbrace{l^1_{41}, l^1_{42}, l^1_{43}}_{\mathbf{u}_1}, \underbrace{l^1_{23}, l^1_{31}, l^1_{12}}_{\mathbf{v}_1}),\ \tilde{\mathbf{l}}^2 = (\underbrace{l^2_{41}, l^2_{42}, l^2_{43}}_{\mathbf{u}_2}, \underbrace{l^2_{23}, l^2_{31}, l^2_{12}}_{\mathbf{v}_2})$ through points $\mathbf{a}_1, \mathbf{b}_1$ and $\mathbf{a}_2, \mathbf{b}_2$ it holds:
  - $\tilde{\mathbf{l}}^1$ and $\tilde{\mathbf{l}}^2$ are coplanar (intersect) iff, the four points are coplanar, i.e. iff

$$\begin{aligned}\det(\tilde{\mathbf{a}}_1, \tilde{\mathbf{b}}_1, \tilde{\mathbf{a}}_2, \tilde{\mathbf{b}}_2) &= l^1_{41} l^2_{23} + l^1_{42} l^2_{31} + l^1_{43} l^2_{12} + l^1_{23} l^2_{41} + l^1_{31} l^2_{42} + l^1_{12} l^2_{43} \\ &= \mathbf{u}_1 \mathbf{v}_2 + \mathbf{u}_2 \mathbf{v}_1 = 0\end{aligned}$$

  - If $\mathbf{u}_1 \mathbf{v}_1 + \mathbf{u}_2 \mathbf{v}_2 < 0$ then $\tilde{\mathbf{l}}^1$ passes $\tilde{\mathbf{l}}^2$ counter clockwise
  - If $\mathbf{u}_1 \mathbf{v}_1 + \mathbf{u}_2 \mathbf{v}_2 > 0$ then $\tilde{\mathbf{l}}^1$ passes $\tilde{\mathbf{l}}^2$ counter clockwise

Euclidean Transformations in $\mathbb{P}^3$

- Euclidean transformations in $\mathbb{P}^3$ have 6 DOF
  - Translation around the vector $\mathbf{T} = (t_x, t_y, t_z)^\top$, with homogeneous transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

  - (Euler) Rotations around the x, y, z-Axis, i.e. $\mathbf{R} = \mathbf{R}_z \mathbf{R}_y \mathbf{R}_x$ with homogeneous transformation matrix

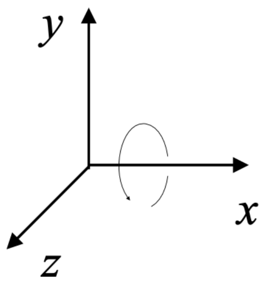
$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- General homogeneous representation of Euclidean transformations

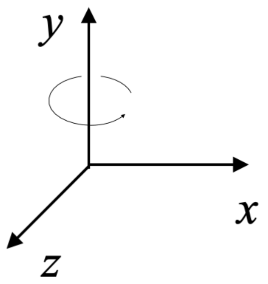
$$\tilde{\mathbf{x}}' = \mathbf{H}_E \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix} \tilde{\mathbf{x}} \quad \text{with} \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I} \quad \text{and} \quad \det(\mathbf{R}) = 1$$

Rotations in $\mathbb{R}^3$

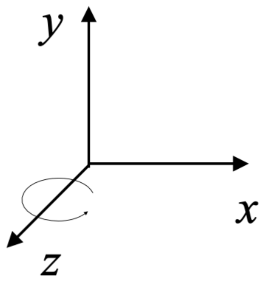
- (Right-handed) rotation around
  - the x-axis $\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$



  - the y-axis $\mathbf{R}_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$



  - the z-axis $\mathbf{R}_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\beta & 0 & 0 \\ \sin\gamma & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$



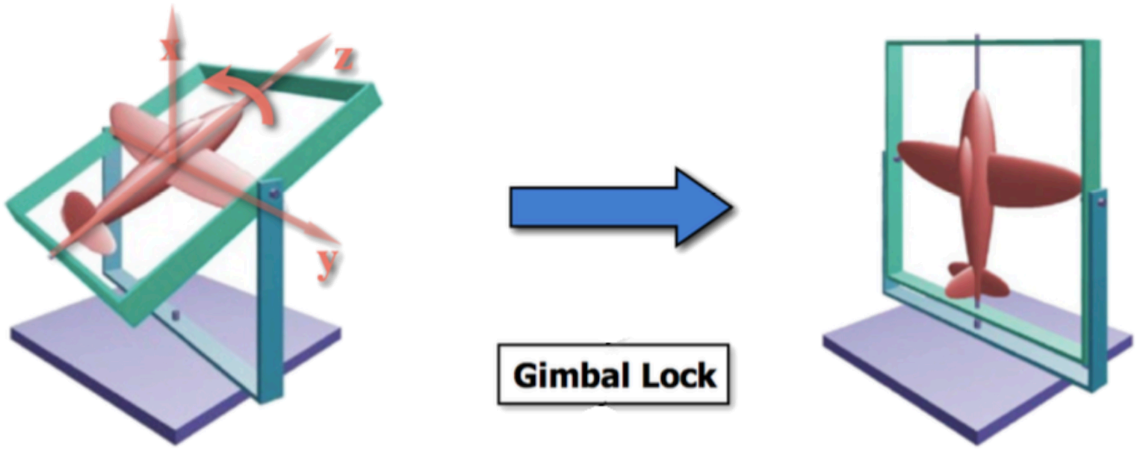
Rotations in $\mathbb{R}^3$ at a given Point $\mathbf{p}$

1. *Translate* object such that the point $\mathbf{p}$ goes to the origin (using matrix $\mathbf{T}$)
2. *Rotate* object (using matrix $\mathbf{R}$)
3. *Translate* object *back* to point $\mathbf{p}$ (using matrix $\mathbf{T}^{-1}$)

$$\mathbf{M} = \mathbf{T}^{-1} \cdot \mathbf{R} \cdot \mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

Euler Rotations

- Rotations around the x, y, z axis are called *Euler Rotations* and α, β, γ *Euler Angles*
- *Drawbacks* of Euler Rotations
  - Order matters
    $R(\alpha) \cdot R(\beta) \cdot R(\gamma) \neq R(\gamma) \cdot R(\beta) \cdot R(\alpha)$
  - Not unique, i.e.
    $R(\alpha) \cdot R(\beta) \cdot R(\gamma) = R(\alpha \pm \pi) \cdot R(\beta \pm \pi) \cdot R(\gamma \pm \pi)$
  - Gimbal Lock
    - Rotations of $\pi/2$ can lead to interference of two axis
  - Unsuitable for animation and optimization
    - No meaningful interpolation of two orientations



Angle-Axis Rotations

- Instead of three angles give a (normalized) axis vector $\mathbf{n} = (x, y, z)^T$ and an angle φ to describe a rotation

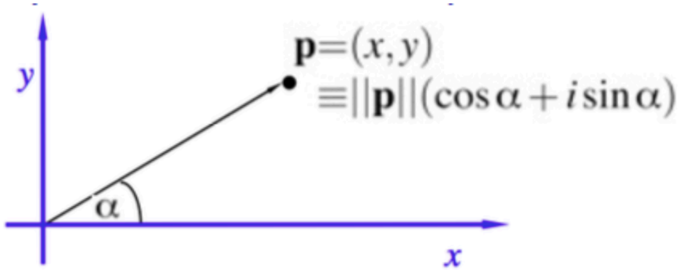
$$
\begin{aligned}
\tilde{\mathbf{R}}(\mathbf{n}, \varphi) &= \tilde{\mathbf{R}}_1^{-1} \cdot \tilde{\mathbf{R}}_2^{-1} \cdot \tilde{\mathbf{R}}_3 \cdot \tilde{\mathbf{R}}_2 \cdot \tilde{\mathbf{R}}_1 \\
&= \begin{pmatrix}
x^2(1-\cos\varphi)+\cos\varphi & xy(1-\cos\varphi)-z\sin\varphi & xz(1-\cos\varphi)+y\sin\varphi & 0 \\
xy(1-\cos\varphi)+z\sin\varphi & y^2(1-\cos\varphi)+\cos\varphi & yz(1-\cos\varphi)-x\sin\varphi & 0 \\
xz(1-\cos\varphi)-y\sin\varphi & yz(1-\cos\varphi)+x\sin\varphi & z^2(1-\cos\varphi)+\cos\varphi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

  - Standard description for a rotation used in computer graphics
  - No gimbal lock
  - **Still not suitable for animations (meaningful interpolation between orientations not possible)**

Some Recap of (Rotation with) Complex Numbers

- A point $\mathbf{p} = (x, y)^T$ in the plane can be represented (using polar coordinates) as a complex number

$$\mathbf{p} = \|\mathbf{p}\|(\cos\alpha + i\sin\alpha) = \|\mathbf{p}\|e^{i\alpha}$$

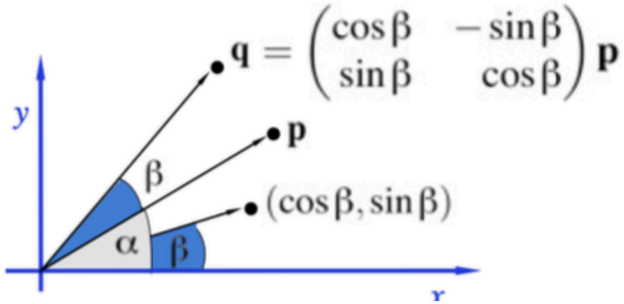


- Multiplication of $\|\mathbf{p}\|e^{i\alpha}$ with $e^{i\beta}$ results in

$$\mathbf{q} = \|\mathbf{p}\|e^{i\alpha} \cdot e^{i\beta} = \|\mathbf{p}\|e^{i(\alpha+\beta)} = \|\mathbf{p}\|(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

With $x = \|\mathbf{p}\|\cos\alpha,\ y = \|\mathbf{p}\|\sin\alpha$ this can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

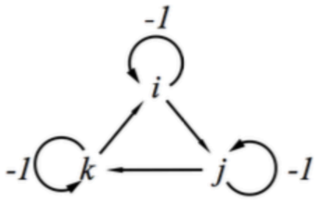


Quaternions

- A quaternion

$$\mathbf{q} = \underbrace{s}_{\text{real}} \cdot 1 + \underbrace{x}_{\text{imaginary}} \cdot i + \underbrace{y}_{\text{imaginary}} \cdot j + \underbrace{z}_{\text{imaginary}} \cdot k = \left[s, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] = [s, \mathbf{v}]$$

with $i^2 = j^2 = k^2 = ijk =$ -



W. R. Hamilton
1805-1865

is the three-dimensional generalisation of a complex number

- Discovered 1853 by W. R. Hamilton

Some important Properties of Quaternions

- *Addition* (associative, commutative, neutral element is $\mathbf{0} = [0, (0, 0, 0)]$)
  $\mathbf{q}_1 + \mathbf{q}_2 = [s_1 + s_2, \mathbf{v}_1 + \mathbf{v}_2]$
- *Multiplication* (**not commutative**, neutral element is $\mathbf{1} = [1, (0, 0, 0)]$)
  $\mathbf{q}_1 \cdot \mathbf{q}_2 = [s_1 \cdot s_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle,\ s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$
- *Distributive*
  $\mathbf{q}(\mathbf{r} + \mathbf{s}) = \mathbf{qr} + \mathbf{qs}$ and $(\mathbf{r} + \mathbf{s})\mathbf{q} = \mathbf{rq} + \mathbf{sq}$
- *Length*
  $\|\mathbf{q}\| = \sqrt{s^2 + x^2 + y^2 + z^2}$
- *Conjugate*
  $\bar{\mathbf{q}} = [s, -\mathbf{v}]$ with $\mathbf{q} \cdot \bar{\mathbf{q}} = s^2 + \|\mathbf{v}\|^2 = \|\mathbf{q}\|^2$

- *Inverse*
  $\mathbf{q}^{-1} = \|\mathbf{q}\|^{-2} \cdot \bar{\mathbf{q}} \quad (\|\mathbf{q}\| = 1 \implies \mathbf{q}^{-1} = \bar{\mathbf{q}})$

Quaternions and Rotation

- A rotation around the axis $\mathbf{v}$ and the the angle φ can be described by a unit quaternion

$$\mathbf{q} = \left[\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2} \cdot \mathbf{v}\right], \ \|\mathbf{q}\| = 1$$

- Rotating a point $\mathbf{p}$ with a quaternion $\mathbf{q}$ is done by

$$R(\mathbf{p}) = \mathbf{q} \cdot \mathbf{p}_q \cdot \bar{\mathbf{q}} \ \text{ with } \ \mathbf{p}_q = [0, \mathbf{p}]$$

- Quaternions $\mathbf{q}$ and $-\mathbf{q}$ (opposite direction and angle) describe the same roation, as

$$-\mathbf{q} \cdot \mathbf{p}_q \cdot \overline{-\mathbf{q}} = -1 \cdot \mathbf{q} \cdot \mathbf{p}_q \cdot \overline{(-1 \cdot \mathbf{q})} = -1 \cdot \mathbf{q} \cdot \mathbf{p}_q \cdot -1 \cdot \bar{\mathbf{q}} = \mathbf{q} \cdot \mathbf{p}_q \cdot \bar{\mathbf{q}}$$

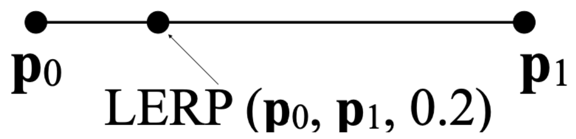
- Concatenation of rotations

$$R_2(R_1(\mathbf{p})) = \mathbf{q}_2 \cdot (\mathbf{q}_1 \cdot \mathbf{p}_q \cdot \bar{\mathbf{q}}_1) \cdot \bar{\mathbf{q}}_2 = (\mathbf{q}_2 \cdot \mathbf{q}_1) \cdot \mathbf{p}_q \cdot \overline{(\mathbf{q}_1 \cdot \mathbf{q}_2)}$$

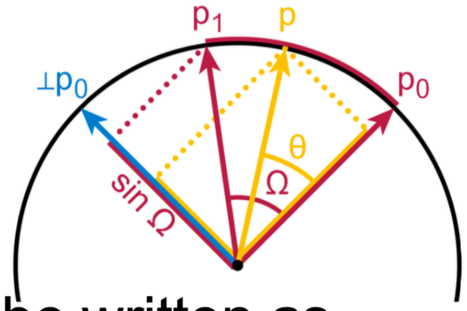
SLERP (Spherical Linear intERPolation)

- Linear interpolation (LERP)
  - Given starting and end point $\mathbf{p}_0$, $\mathbf{p}_1$, interpolation parameter $t \in [0,1]$

$$\mathrm{LERP}(\mathbf{p}_0, \mathbf{p}_1, t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$



- Spherical linear interpolation (SLERP)
  - LERP on the surface of a (unit) sphere



$$\mathrm{SLERP}(\mathbf{p}_0, \mathbf{p}_1, t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}\mathbf{p}_0 + \frac{\sin[t\Omega]}{\sin\Omega}\mathbf{p}_1$$

Using quaternions $\mathbf{q} = \left[\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2}\mathbf{v}\right]$ this can be written as

$$\mathrm{SLERP}(\mathbf{q}_0, \mathbf{q}_1, t) = \mathbf{q}_0\left(\mathbf{q}_0^{-1}\mathbf{q}_1\right)^t = \mathbf{q}_0 \exp\left(t \cdot \log\left(\mathbf{q}_0^{-1}\mathbf{q}_1\right)\right)$$ with

$$\log(\mathbf{q}) = \left[0, \frac{\varphi}{2}\mathbf{v}\right], \ \exp\left(\left[0, \frac{\varphi}{2}\mathbf{v}\right]\right) = \left[\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2} \cdot \mathbf{v}\right] = \mathbf{q}$$

Similarity Transformations in $\mathbb{P}^3$

- Euclidean transformations and isotropic scaling (7 DOF)
  - Homogenous matrix vector representation

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \cdot \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

or $\tilde{\mathbf{x}}' = \mathbf{H}_S \tilde{\mathbf{x}} = \begin{bmatrix} s\mathbf{R} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix} \tilde{\mathbf{x}}$ with $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ (and $\det(\mathbf{R}) = 1$)

Special Affine Transformations in $\mathbb{P}^3$

- Similarity transformations and anisotropic scaling (9 DOF)
  - Homogenous matrix vector representation

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \cdot \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Special Affine Transformations in $\mathbb{P}^3$

Affine Transformations in $\mathbb{P}^3$

- General affine transformation (12 DOF)
  - Homogeneous matrix vector representation

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

can be written in the form (see also camera models)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \cdot & [\mathbf{R} \;\; \mathbf{T}] \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

or $\tilde{\mathbf{x}}' = \mathbf{H}_A \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{A} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix} \tilde{\mathbf{x}}$

Projective Transformations in $\mathbb{P}^3$

- General projective transformation (15 DOF)
  - Homogeneous matrix vector representation

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

can be written in the form

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \cdot & [\mathbf{R} \;\; \mathbf{T}] \\ p_{41} \quad p_{42} \quad p_{43} & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

or $\tilde{\mathbf{x}}' = \mathbf{H}\tilde{\mathbf{x}} = \begin{bmatrix} & \mathbf{A} & & \mathbf{T} \\ p_{41} & p_{42} & p_{43} & 1 \end{bmatrix} \tilde{\mathbf{x}}$

Overview of 3D Transformations

Transformation	Matrix	# DOF	Preserves
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3\times4}$	3	orientation
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times4}$	6	length
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times4}$	7	angles
affine	$\begin{bmatrix} \mathbf{A} & \mathbf{t} \end{bmatrix}_{3\times4}$	12	parallelism
projective	$\begin{bmatrix} \mathbf{H} \end{bmatrix}_{4\times4}$	15	straight lines

- 3D transformations are defined analogously to 2D transformations
- 3×4 matrices are extended with a fourth $[\mathbf{0}^\top 1]$ row for homogeneous transforms

Transformations on Co-Vectors

If a point $\tilde{\mathbf{x}}$ is transformed by a perspective 2D (3D) transformation $\tilde{\mathbf{H}}$ as

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$$

for a transformed 2D line (3D Plane) it must hold

$$0 = \tilde{\mathbf{l}}'^\top \tilde{\mathbf{x}}' = \tilde{\mathbf{l}}'^\top \tilde{\mathbf{H}}\tilde{\mathbf{x}} = (\tilde{\mathbf{H}}^\top \tilde{\mathbf{l}}')^\top \tilde{\mathbf{x}} = \tilde{\mathbf{l}}^\top \tilde{\mathbf{x}}$$

and therefore

$$\tilde{\mathbf{l}}' = \tilde{\mathbf{H}}^{-\top} \tilde{\mathbf{l}}$$

Thus, the **action of a projective transformation on a co-vector** such as a 2D line or 3D plane can be represented by the **transposed inverse** of the matrix.

The transformation on a 3D line $\tilde{\mathbf{L}}$ is given by $\tilde{\mathbf{H}}\tilde{\mathbf{L}}\tilde{\mathbf{H}}^\top$

Plane at infinity

- If points on $\tilde{\mathbf{m}}_\infty = (0,0,0,1)^\top$ are mapped by an affine transformation $\tilde{\mathbf{H}}$ it holds

$$\tilde{\mathbf{m}}'_\infty = (\tilde{\mathbf{H}}^{-1})^\top \tilde{\mathbf{m}}_\infty = \begin{bmatrix} \mathbf{A}^{-\top} & 0 \\ -\mathbf{T}^\top \mathbf{A}^{-\top} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \tilde{\mathbf{m}}_\infty$$

- The plane at infinity $\tilde{\mathbf{m}}_\infty = (0,0,0,1)^\top$ is fixed under $\mathbf{H}$, iff $\mathbf{H}$ is an **affine transform**
- Properties of the plane at infinity
  - Canonical position $\tilde{\mathbf{m}}_\infty = (0,0,0,1)^\top$
  - Contains the directions (vanishing points)
  - Two planes are parallel iff their intersection line $\tilde{\mathbf{l}}_\infty = (x_1, x_2, x_3, 0)^\top$ is in $\tilde{\mathbf{m}}_\infty$

- A straight line is parallel to a straight line (or a plane) iff its intersection is in $\tilde{\mathbf{m}}_\infty$