3D Computer Vision (SoSe2024) *Primitives and Transformations in 3D*

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Last lecture

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- 1D Transformations
- Primitives and Transformations
 - Homogeneous Coordinates
 - Points, Lines and Planes
 - 2D Transformations
 - Homography Estimation

Today's Lecture

Today's Lecture

- The 3D Projective Space \mathbb{P}^3
 - Points, Planes,
 - Straight Lines, ...
 - ... and their transformations

Reminder

Points

Points in 1D/2D/3D can be written in **inhomogeneous coordinates** as

$$x\in \mathbb{R}$$
 or $\mathbf{x}=inom{x}{y}\in \mathbb{R}^2$ or $\mathbf{x}=inom{x}{y}{z}\in \mathbb{R}^3$

or in homogenous coordinates as

$$ilde{\mathbf{x}} = egin{pmatrix} ilde{x} \ ilde{w} \end{pmatrix} \in \mathbb{P}^1 \quad ext{or} \quad ilde{\mathbf{x}} = egin{pmatrix} ilde{x} \ ilde{y} \ ilde{w} \end{pmatrix} \in \mathbb{P}^2 \quad ext{or} \quad ilde{\mathbf{x}} = egin{pmatrix} ilde{x} \ ilde{y} \ ilde{z} \ ilde{w} \end{pmatrix} \in \mathbb{P}^3$$

where $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ is called **projective space**. Homogeneous vectors that differ

only by scale are considered equivalent and define an equivalence class, thus homogeneous vectors are **defined only up to scale**.

Points

An inhomogeneous vector \mathbf{x} is converted to a homogeneous vector $\mathbf{\tilde{x}}$ as follows

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix} = \bar{x} \quad \text{or} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \bar{\mathbf{x}} \quad \text{or} \quad \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \bar{\mathbf{x}}$$

with the **augmented vector** $\bar{\mathbf{x}}$. To convert in opposite direction one has to divide by \tilde{w} :

$$ar{\mathbf{x}} = egin{pmatrix} \mathbf{x} \ 1 \end{pmatrix} = egin{pmatrix} x \ y \ z \ 1 \end{pmatrix} = rac{1}{ ilde{w}} \mathbf{ ilde{x}} = rac{1}{ ilde{w}} egin{pmatrix} ilde{x} \ ilde{y} \ ilde{z} \ ilde{w} \end{pmatrix} = egin{pmatrix} ilde{x}/ ilde{w} \ ilde{z}/ ilde{w} \ 1 \end{pmatrix}$$

Homogeneous points whose last element is $\tilde{w} = 0$ can't be represented with inhomogeneous coordinates. They are called **ideal points** or **points at infinity**.

2D Lines

2D lines can also be expressed using homogeneous coordinates $\mathbf{ ilde{l}} = (a,b,c)^ op$

$$\{ ilde{\mathbf{x}} \mid ilde{\mathbf{l}} ilde{\mathbf{x}} = 0\} \quad \Leftrightarrow \quad \{x,y \mid ax+by+c = 0\}$$

- We can normalize $\mathbf{\tilde{l}}$ to $\mathbf{\tilde{l}}=(n_x,n_y,d)^{ op}=(\mathbf{n},d)^{ op}$ with $\|\mathbf{n}\|_2=1$ (Hesse normal form)
 - \circ In this case, **n** is the **normal vector** perpendicular to the line and d is its **distance** to the origin.
- An exception is the line at infinity $\tilde{\mathbf{l}}_{\infty} = (0,0,1)^{ op}$ which passes through all ideal points.

Duality (in 2D)

The duality principle:

For every proposition of two-dimensional projective geometry there exists a **dual proposition** which is obtained by interchanging the role of points and lines in the original proposition.

• For example, dual are

$$egin{array}{cccc} ilde{\mathbf{x}} & \longleftrightarrow & ilde{\mathbf{l}} & \ ilde{\mathbf{x}}^T ilde{\mathbf{l}} = 0 & \longleftrightarrow & ilde{\mathbf{l}}^T ilde{\mathbf{x}} = 0 & \ ilde{\mathbf{x}} = ilde{\mathbf{l}} imes ilde{\mathbf{l}}' & \longleftrightarrow & ilde{\mathbf{l}} = ilde{\mathbf{x}} imes ilde{\mathbf{x}}' & \ \end{array}$$

Transformation	Matrix	# DOF	Preserves
translation	$[\mathbf{I} \mathbf{t}]_{2 imes 3}$	2	orientation
rigid (Euclidean)	$[\mathbf{R} \mathbf{t}]_{2 imes 3}$	3	length
similarity	$[s{f R} {f t}]_{2 imes 3}$	4	angles
affine	$[\mathbf{A} \mathbf{t}]_{2 imes 3}$	6	parallelism
projective	$[\mathbf{H}]_{3 imes 3}$	8	straight lines

- Transformations form **nested set of groups** (closed under composition, inverse)
- 2 imes 3 matrices are extended with a third $[\mathbf{0}^ op \mathbf{1}]$ row for homogeneous transforms

The Projective Space \mathbb{P}^3

3D Planes

3D planes can also be represented with homogeneous coordinates $ilde{\mathbf{m}} = (a, b, c, d)^ op$ as

$$\{ ilde{\mathbf{x}} \mid ilde{\mathbf{m}}^{ op} ar{\mathbf{x}} = 0\} \quad \Leftrightarrow \quad \{x,y,z \mid ax+by+cz+d=0\}$$

- $\tilde{\mathbf{m}}$ can be **normalized**: $\tilde{\mathbf{m}} = (n_x, n_y, n_z, d)^\top = (\mathbf{n}, d)^\top$, $\|\mathbf{n}\|_2 = 1$ (Hesse normal form)
 - \circ **n** is the **normal vector** the plane and *d* its **distance** to the origin



• An exception is the **plane at infinity** $\tilde{\mathbf{m}} = (0, 0, 0, 1)^{\top}$ which passes through all ideal points (=points at infinity) for which $\tilde{w} = 0$

Plane given by Three Points in \mathbb{P}^3

- For a plane $\tilde{\mathbf{m}}$ that contains the three points $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3$ (in general position) it must hold $\tilde{\mathbf{x}}_1^\top \tilde{\mathbf{m}} = \tilde{\mathbf{x}}_2^\top \tilde{\mathbf{m}} = \tilde{\mathbf{x}}_3^\top \tilde{\mathbf{m}} = 0$ or $\mathbb{R}^{3 \times 4}$ $\begin{bmatrix} \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \\ \tilde{\mathbf{x}}_3^\top \end{bmatrix} \stackrel{\mathbb{Q}}{\overset{\mathbb{Q}}{\overset{\mathbb{Q}}{\mathbf{m}}}} = 0$
- Thus the plane $\tilde{\mathbf{m}}$ that contains the three points $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3$ is the **right null space** of

$$egin{bmatrix} ilde{\mathbf{x}}_1^ op \ ilde{\mathbf{x}}_2^ op \ ilde{\mathbf{x}}_3^ op \end{bmatrix} = egin{pmatrix} x_{1_1} & x_{1_2} & x_{1_3} & x_{1_4} \ x_{2_1} & x_{2_2} & x_{2_3} & x_{2_4} \ x_{3_1} & x_{3_2} & x_{3_3} & x_{3_4} \end{pmatrix}$$

• Does that relate to calculating the straight line through two points?

Reminder: Straight Line given by Two Points in \mathbb{P}^2

• A line $\tilde{\mathbf{l}}$ in \mathbb{P}^2 through $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2$ (in general position) is given by the **right null space** of

$$\begin{bmatrix} \tilde{\mathbf{x}}_1^\top \\ \tilde{\mathbf{x}}_2^\top \end{bmatrix} = \begin{pmatrix} x_{1_1} & x_{1_2} & x_{1_3} \\ x_{2_1} & x_{2_2} & x_{2_3} \end{pmatrix} \begin{pmatrix} \ell_{\mathcal{A}} \\ \ell_{\mathcal{A}} \\ \ell_{\mathcal{A}} \end{pmatrix} = \begin{pmatrix} \mathfrak{S} \\ \mathfrak{S} \end{pmatrix}$$

which can be calculated directly by the cross product, i.e. $\tilde{\mathbf{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$

• As any point $ilde{\mathbf{x}}$ on the line $ilde{\mathbf{l}}$ is a linear combination of $ilde{\mathbf{x}}_1, ilde{\mathbf{x}}_2$ it must hold

$$\det egin{bmatrix} \mathbf{ ilde{x}}^{ op} \ \mathbf{ ilde{x}}^{ op} \ \mathbf{ ilde{x}}^{ op} \ \mathbf{ ilde{x}}^{ op} \ \mathbf{ ilde{x}}^{ op} \end{bmatrix} = \det egin{pmatrix} x_1 & x_2 & x_3 \ x_{1_1} & x_{1_2} & x_{1_3} \ x_{2_1} & x_{2_2} & x_{2_3} \end{pmatrix} = 0$$

• Laplace expansion results in $0 = x_1 D_{23} - x_2 D_{13} + x_3 D_{12}$ with

$$D_{ij}=\detegin{pmatrix} x_{1_i} & x_{1_j} \ x_{2_i} & x_{2_j} \end{pmatrix}$$
, i.e. $ilde{\mathbf{l}}=(D_{23},-D_{13},D_{12})^ op$

Plane given by Three Points in \mathbb{P}^3

• As any point $\tilde{\mathbf{x}}$ on the plane $\tilde{\mathbf{m}}$ is a linear combination of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3$, it must hold

$$\det \begin{bmatrix} \tilde{\mathbf{x}}^{\top} \\ \tilde{\mathbf{x}}_{1}^{\top} \\ \tilde{\mathbf{x}}_{2}^{\top} \\ \tilde{\mathbf{x}}_{3}^{\top} \end{bmatrix} = \det \begin{pmatrix} x_{1} & x_{2} & x_{3} & x_{4} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix} = 0$$

• Laplace expansion results in

$$0=x_1D_{234}-x_2D_{134}+x_3D_{124}+x_4D_{123}$$
 with $D_{ijk}=\detegin{pmatrix} x_{1_i}&x_{1_k}\ x_{2_i}&x_{2_j}&x_{2_k}\ x_{3_i}&x_{3_j}&x_{3_k} \end{pmatrix}$

and thus
$$ilde{\mathbf{m}} = (D_{234}, -D_{134}, D_{124}, -D_{123})^ op$$

Example: Plane given by Three non-vanishing Points in \mathbb{P}^3

- For the plane $\mathbf{\tilde{m}}$ through the three non-vanishing points

$$ilde{\mathbf{x}}_1 = egin{pmatrix} \mathbf{x}_1 \ 1 \end{pmatrix}, \quad ilde{\mathbf{x}}_2 = egin{pmatrix} \mathbf{x}_2 \ 1 \end{pmatrix}, \quad ilde{\mathbf{x}}_3 = egin{pmatrix} \mathbf{x}_3 \ 1 \end{pmatrix}$$

it holds e.g.

$$D_{234}=\detegin{pmatrix} x_{1_2}&x_{1_3}&1\ x_{2_2}&x_{2_3}&1\ x_{3_2}&x_{3_3}&1 \end{pmatrix}=\detegin{pmatrix} x_{1_2}-x_{3_2}&x_{1_3}-x_{3_3}&0\ x_{2_2}-x_{3_2}&x_{2_3}-x_{3_3}&0\ x_{3_2}&x_{3_3}&1 \end{pmatrix}=((\mathbf{x}_1-\mathbf{x}_3) imes(\mathbf{x}_2-\mathbf{x}_3))_1$$

and thus

$$ilde{\mathbf{m}} = egin{bmatrix} (\mathbf{x}_1 - \mathbf{x}_3) imes (\mathbf{x}_2 - \mathbf{x}_3) \ -\mathbf{x}_3^ op (\mathbf{x}_1 imes \mathbf{x}_2) \end{bmatrix}$$

Duality (in 3D)

The duality principle:

For every proposition of three-dimensional projective geometry there exists a **dual proposition** which is obtained by interchanging the role of points and planes in the original proposition.

• Dual are

 $egin{array}{ccc} ilde{{f x}} & \longleftrightarrow & ilde{{f m}} \ ilde{{f m}} & = 0 & \longleftrightarrow & ilde{{f m}}^T ilde{{f x}} = 0 \end{array}$

Straight Lines are dual to Straight Lines

Straight Lines in Projective Space \mathbb{P}^3

- Representing a straight line in 3D is problematic
 - Straight lines in 3D have 4 DOF Ο
 - Thus they would have to be represented by a homogeneous vector with five elergents 0
 - Are there more suitable representations?
- Usual representations for straight lines in space(+) = $a + 1 \cdot b$ are 622+62+62=1
 - Linear combination of two points Ο
 - **Plücker matrices** Ο
 - Plücker coordinates 0



Straight Lines in \mathbb{P}^3 as Linear Combination of Points

A straight line l can be represented as the space spanned by the matrix W, consisting of points (on a straight line) a and b

$$ilde{\mathbf{W}} = egin{bmatrix} ilde{\mathbf{a}}^{ op} \ ilde{\mathbf{b}}^{ op} \end{bmatrix} \mathcal{E}^{2 imes 4}$$

- The **span** of $\tilde{\mathbf{W}}^{ op}$ ist the bundle of points $\lambda \tilde{\mathbf{a}} + \mu \tilde{\mathbf{b}}$ on the straight line \mathbf{l}
- $\circ~$ The span of the 2-dimensional right null space of \tilde{W} is the bundle of planes with the straight line l as axis



Straight Lines in \mathbb{P}^3 as Linear Combination of Planes

• The **dual representation** as the right null space of the matrix

$$ilde{\mathbf{W}}^* = egin{bmatrix} ilde{\mathbf{p}}^{ op} \ ilde{\mathbf{q}}^{ op} \end{bmatrix}$$

contains the planes ${\boldsymbol{p}}$ and ${\boldsymbol{q}},$ that intersect in the straight line ${\boldsymbol{l}}$

- The **span** of $\mathbf{\tilde{W}^*}^{ op}$ ist the bundle of planes $\lambda' \mathbf{\tilde{p}} + \mu' \mathbf{\tilde{q}}$ with the axis \mathbf{l}
- $\circ~$ The span of the 2-dimensional right null space of \tilde{W}^* is the bundle of points on the straight line l as axis



• Relationship between the two s representations: $\tilde{\mathbf{W}}^*\tilde{\mathbf{W}}^\top = \tilde{\mathbf{W}}\tilde{\mathbf{W}}^*^\top = \mathbf{0}_{2\times 2}$

Example

• The *x*-axis ...

 \circ ... is spannend by the point $\mathbf{a} = (0,0,0)^ op$ and the direction of the x-axis $\mathbf{b} = (1,0,0)^ op$ and thus

$$ilde{\mathbf{W}} = egin{pmatrix} 0 & 0 & 0 & 1 \ 1 & 0 & 0 & 0 \end{pmatrix}$$

• ... is the right null space of the xy-plane $\tilde{\mathbf{p}} = (0, 0, 1, 0)^{\top}$ and the xz-plane $\tilde{\mathbf{p}} = (0, 1, 0, 0)^{+}$ and thus

(1 | & | 0

Union and Intersection of Points, Lines and Planes

ER^{3X4}

• The plane $\tilde{\mathbf{m}}$, given by the union of the straight line $\tilde{\mathbf{W}}$ with the point $\tilde{\mathbf{x}}$ is the right null space of the matrix $\tilde{\mathbf{M}}$, containing $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{x}}$, i.e.

$$\tilde{\mathbf{M}}\tilde{\mathbf{m}} = \mathbf{0}$$
 with $\tilde{\mathbf{M}} = \begin{bmatrix} \tilde{\mathbf{W}} \\ \tilde{\mathbf{x}}^{\top} \end{bmatrix}$

• The point $\tilde{\mathbf{x}}$, given by the intersection of the plane $\tilde{\mathbf{m}}$ and the straight line $\tilde{\mathbf{W}}^*$ is the right null space of the matrix $\tilde{\mathbf{M}}$, containing $\tilde{\mathbf{W}}^*$ and $\tilde{\mathbf{m}}$, i.e.

$$ilde{\mathbf{M}} ilde{\mathbf{x}} = \mathbf{0}$$
 with $ilde{\mathbf{M}} = \begin{bmatrix} ilde{\mathbf{W}}^* \\ ilde{\mathbf{m}}^\top \end{bmatrix}$





Example

- The plane $\tilde{\mathbf{m}}$, given by the union of the straight line $\tilde{\mathbf{W}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and the point $(0,0,1)^{\top}$ is the right null space of $\tilde{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ which results in $\tilde{\mathbf{m}} = (0,1,0,0)^{\top}$
- The point $\tilde{\mathbf{p}}$, given by the intersection of the plane z = 1 and the straight line $\tilde{\mathbf{W}}^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is the right null space of

$$\tilde{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{which results in } \tilde{\mathbf{p}} = (1, 0, 0, 0)^{\top}$$

Plücker Coordinates

- For the line given by the two points \mathbf{a}, \mathbf{b} it holds $\tilde{\mathbf{W}} = \left| \begin{array}{c} \tilde{\mathbf{a}}^\top \\ \tilde{\mathbf{b}}^\top \end{array} \right|$, rank $(\tilde{\mathbf{W}}) = 2$
- For two other points \mathbf{a}', \mathbf{b}' on the same line \mathbf{l} it holds

$$\tilde{\mathbf{W}}' = \begin{bmatrix} \tilde{\mathbf{a}'}^{\top} \\ \mathbf{b'}^{\top} \end{bmatrix}_{\mathbf{v}, \mathbf{v}}^{\mathbf{v}, \mathbf{v}} \in \mathcal{X}_{n}, \mathbf{v} \in \mathcal{X}_{n}, \mathbf{v}}^{\mathbf{v}, \mathbf{v}, \mathbf{v}} = \mathbf{1} \text{ and } \tilde{\mathbf{W}}' = \mathbf{\Lambda} (\mathbf{W})^{\mathbf{v}} \text{ with } \mathbf{\Lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \text{ det}(\mathbf{\Lambda}) \neq \mathbf{0}$$

• In particular we get

$$egin{pmatrix} a'_i & b'_i \ a'_j & b'_j \end{pmatrix} = egin{pmatrix} \lambda_{1,1} & \lambda_{1,2} \ \lambda_{2,1} & \lambda_{2,2} \end{pmatrix} egin{pmatrix} a_i & b_i \ a_j & b_j \end{pmatrix} \implies \det egin{pmatrix} a'_i & b'_i \ a'_j & b'_j \end{pmatrix} = \det \Lambda \cdot \underbrace{\det egin{pmatrix} a_i & b_i \ a_j & b_j \end{pmatrix}}_{=:l_{ij}}, \hspace{0.2cm} i,j=1,\ldots,4$$

• As $l_{ii} = 0$, $\binom{4}{2} = 6$ independent quantities remain: $(l_{12}: l_{13}: l_{14}: l_{23}: l_{24}: l_{34})$

Plücker Matrix

• The **Plücker Matrix** $\tilde{\mathbf{L}}$ describes the straight line through the two points \mathbf{a}, \mathbf{b} :

$$ilde{\mathbf{L}} = ilde{\mathbf{a}} ilde{\mathbf{b}}^ op - ilde{\mathbf{b}} ilde{\mathbf{a}}^ op = egin{pmatrix} 0 & l_{12} & l_{13} & l_{14} \ -l_{12} & 0 & l_{23} & l_{24} \ -l_{13} & -l_{23} & 0 & l_{34} \ -l_{14} & -l_{24} & -l_{34} & 0 \end{pmatrix}, \ i. \, e. \ \ l_{ij} = a_i b_j - b_i a_j$$

• Example: Representing the x-axis as Plücker matrix, defined by the two points $\tilde{\mathbf{a}} = (0, 0, 0, 1)^{\top}$ and $\tilde{\mathbf{b}} = (1, 0, 0, 0)^{\top}$:

$$ilde{\mathbf{L}} = egin{pmatrix} 0 \ 0 \ 0 \ 1 \end{pmatrix} (1,0,0,0) - egin{pmatrix} 1 \ 0 \ 0 \ 0 \end{pmatrix} (0,0,0,1) = egin{pmatrix} 0 & 0 & 0 & -1 \ 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 \ 1 & 0 & 0 & 0 \ 1 & 0 & 0 & 0 \end{pmatrix}$$

Properties of Plücker Matrices

- $ilde{\mathbf{L}}$ is a skew symmetric homogeneous 4 imes 4 Matrix
- $\mathsf{rank}(\mathbf{L}) = 2$
 - $\circ~2$ -dim null space is the bundle of planes defined by the straight line (it holds $ilde{f L} ilde{f W}^{*}^{ op}=f 0_{4 imes 2}$)
- 4 DOF
 - $\circ~$ 6 independet entries homogenity det($\mathbf{\tilde{L}}=\mathbf{0}$)
- $\tilde{\mathbf{L}} = \tilde{\mathbf{a}}\tilde{\mathbf{b}}^{\top} \tilde{\mathbf{b}}\tilde{\mathbf{a}}^{\top}$
 - gereralization of the crossproduct to 4D space (6 subdeterminants of $[{f ilde a},{f ilde b}]$)
- $\tilde{\mathbf{L}}$ is independent of the choice of \mathbf{a}, \mathbf{b} on the line, as for any $\tilde{\mathbf{c}} = \tilde{\mathbf{a}} + \mu \tilde{\mathbf{b}}$ it holds

$$ilde{\mathbf{L}}' = ilde{\mathbf{a}} ilde{\mathbf{c}}^ op - ilde{\mathbf{c}} ilde{\mathbf{a}}^ op = ilde{\mathbf{a}}(ilde{\mathbf{a}}^ op + \mu ilde{\mathbf{b}}^ op) - (ilde{\mathbf{a}} + \mu ilde{\mathbf{b}}) ilde{\mathbf{a}}^ op = ilde{\mathbf{a}} ilde{\mathbf{b}}^ op - ilde{\mathbf{b}} ilde{\mathbf{a}}^ op = ilde{\mathbf{L}}$$

• If $\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$ is a point transform, the Plücker Matrix is transformed by $\tilde{\mathbf{L}}' = \tilde{\mathbf{H}}\tilde{\mathbf{L}}\tilde{\mathbf{H}}^{\top}$

Dual Plücker Matrices

- The dual Plücker matrix $\tilde{\mathbf{L}}^* = \tilde{\mathbf{p}} \tilde{\mathbf{q}}^\top \tilde{\mathbf{q}} \tilde{\mathbf{p}}^\top$ describes the intersection of planes $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$
- Transformation ot the dual Plücker matrix $ilde{f L}^*$ with the homography $ilde{f H}$ is given by $(ilde{f H}^{-1})^ op ilde{f L}^* ilde{f H}^{-1}$
- Relationship between the Plücker matrix $\tilde{\mathbf{L}}$ and its dual $\tilde{\mathbf{L}}^*$:

$$l_{12}: l_{13}: l_{13}: l_{23}: l_{42}: l_{34} = l_{34}^*: l_{42}^*: l_{23}^*: l_{14}^*: l_{13}^*: l_{12}^*$$

- Union and Incidence
 - $\circ~$ Plane through point and straight line: $\mathbf{\tilde{m}}=\mathbf{\tilde{L}}^{*}\mathbf{\tilde{x}}$
 - Point on straight line: $\mathbf{\tilde{L}}^*\mathbf{\tilde{x}} = \mathbf{0}$
 - $\circ~$ Intersection between plane and straight line: $\mathbf{\tilde{x}}=\mathbf{\tilde{L}}\mathbf{\tilde{m}}$
 - $\circ~$ Straight line in plane: $\mathbf{\tilde{L}}^{*}\mathbf{\tilde{m}}=\mathbf{0}$
 - $\circ~$ Coplanar straight lines: $[\mathbf{\tilde{L}}_1,\mathbf{\tilde{L}}_2,\dots]\mathbf{\tilde{x}}=\mathbf{0}$

Example: Intersection and the Plücker Matrix

• For the intersection $ilde{\mathbf{x}}$ of the x-axis with the plane x=1 it holds $ilde{\mathbf{x}}= ilde{\mathbf{L}} ilde{\mathbf{m}}$ and thus

$$\tilde{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Plücker Coordinates for non-vanishing Points

- For non-vanishing points \mathbf{a}, \mathbf{b} , i.e. $\tilde{\mathbf{a}} = (a_1, a_2, a_3, 1)^{\top}, \tilde{\mathbf{b}} = (b_1, b_2, b_3, 1)^{\top}$ it holds $(l_{41}, l_{42}, l_{43})^{\top} = \mathbf{b} - \mathbf{a}$ and $(l_{23}, l_{31}, l_{12})^{\top} = \mathbf{a} \times \mathbf{b}$
- The l_{ij} can be interpreted as a homogeneous vector

$$ilde{\mathbf{l}} = (l_{41}, l_{42}, l_{43}, l_{23}, l_{31}, l_{12})^{ op} \in \mathbb{P}^5$$

• They are called **Plücker coordinates** if the condition $\det(L) = 0$ holds, i.e. if $(l_{41}, l_{42}, l_{43})(l_{23}, l_{31}, l_{12})^{\top} = l_{41}l_{23} + l_{42}l_{31} + l_{43}l_{12} = 0$

(only then the vector represents a straight line)

Properties for Plücker Coordinates

- For the two straight lines $\tilde{\mathbf{l}}^1 = (\underbrace{l_{41}^1, l_{42}^1, l_{43}^1}_{\mathbf{u}_1}, \underbrace{l_{23}^1, l_{31}^1, l_{12}^1}_{\mathbf{v}_1}), \ \tilde{\mathbf{l}}^2 = (\underbrace{l_{41}^2, l_{42}^2, l_{43}^2}_{\mathbf{u}_2}, \underbrace{l_{23}^2, l_{31}^2, l_{12}^2}_{\mathbf{v}_2}) \text{ through points } \mathbf{a}_1, \mathbf{b}_1$ and $\mathbf{a}_2, \mathbf{b}_2$ it holds:
 - \circ $ilde{l}^1$ and $ilde{l}^2$ are coplanar (intersect) iff, the four points are coplanar, i.e. iff

$$det(\mathbf{\tilde{a}_1}, \mathbf{\tilde{b}_1}, \mathbf{\tilde{a}_2}, \mathbf{\tilde{b}_2}) = l_{41}^1 l_{23}^2 + l_{42}^1 l_{31}^2 + l_{43}^1 l_{12}^2 + l_{23}^1 l_{41}^2 + l_{31}^1 l_{42}^2 + l_{12}^1 l_{43}^2 \\ = \mathbf{u}_1 \mathbf{v}_2 + \mathbf{u}_2 \mathbf{v}_1 = 0$$

- $\circ~$ If $u_1v_1+u_2v_2<0$ then \tilde{l}^1 passes \tilde{l}^2 counter clockwise
- $\circ~$ If $u_1v_1+u_2v_2>0$ then \tilde{l}^1 passes \tilde{l}^2 counter clockwise

Euclidean Transformations in \mathbb{P}^3

- Euclidean transformations in \mathbb{P}^3 have 6 DOF
 - $\circ~$ Translation around the vector $\mathbf{T}=(t_x,t_y,t_z)^ op$, with homogeneous transformation matrix

$$egin{pmatrix} 1 & 0 & 0 & t_x \ 0 & 1 & 0 & t_y \ 0 & 0 & 1 & t_z \ 0 & 0 & 0 & 1 \end{pmatrix} = egin{pmatrix} \mathbf{I} & \mathbf{T} \ \mathbf{0}^ op & \mathbf{I} \end{bmatrix}$$

• (Euler) Rotations around the x, y, z-Axis, i.e. $\mathbf{R} = \mathbf{R}_z \mathbf{R}_y \mathbf{R}_x$ with homogeneous transformation matrix

$$egin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \ r_{21} & r_{22} & r_{23} & 0 \ r_{31} & r_{32} & r_{33} & 0 \ 0 & 0 & 0 & 1 \end{pmatrix} = egin{pmatrix} \mathbf{R} & \mathbf{0} \ \mathbf{0}^ op & \mathbf{1} \end{bmatrix}$$

• General homogeneous representation of Euclidean transformations

$$\mathbf{\tilde{x}}' = \mathbf{H}_E \mathbf{\tilde{x}} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} \mathbf{\tilde{x}}$$
 with $\mathbf{R}^{\top} \mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$

Rotations in \mathbb{R}^3

(Right-handed) rotation around

→ the x-axis
$$\mathbf{R}_{x}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ the y-axis
$$\mathbf{R}_{y}(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \beta & 0 & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ the z-axis
$$\mathbf{R}_{z}(\gamma) = \begin{pmatrix} \cos \gamma & -\sin \beta & 0 & 0\\ \sin \gamma & \cos \beta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Rotations in \mathbb{R}^3 at a given Point \mathbf{p}

- 1. Translate object such that the point ${f p}$ goes to the origin (using matrix ${f T}$)
- 2. Rotate object (using matrix \mathbf{R})
- 3. *Translate* object *back* to point \mathbf{p} (using matrix \mathbf{T}^{-1})

$$\mathbf{M} = \mathbf{T}^{-1} \cdot \mathbf{R} \cdot \mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{p} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\mathbf{p} \\ \mathbf{0}^{\top} & 1 \end{bmatrix}$$

Euler Rotations

- Rotations around the x,y,z axis are called Euler Rotations and $lpha,eta,\gamma$ Euler Angles
- Drawbacks of Euler Rotations
 - $\circ \quad \text{Order matters} \\ R(\alpha) \cdot R(\beta) \cdot R(\gamma) \neq R(\gamma) \cdot R(\beta) \cdot R(\alpha)$
 - Not unique, i.e. $R(\alpha) \cdot R(\beta) \cdot R(\gamma) = R(\alpha \pm \pi) \cdot R(\beta \pm \pi) \cdot R(\gamma \pm \pi)$
 - Gimbal Lock
 - $\circ~$ Rotations of $\pi/2$ can lead to interference of two axis
 - Unsuitable for animation and optimization
 - \circ $\,$ No meaningful interpolation of two orientations



Angle-Axis Rotations

Instead of three angles give a (normalized) axis vector
 n = (x, y, z)^T and an angle *φ* to describe a rotation

$$\begin{split} \tilde{\mathsf{R}}(\mathsf{n},\varphi) &= \tilde{\mathsf{R}}_1^{-1} \cdot \tilde{\mathsf{R}}_2^{-1} \cdot \tilde{\mathsf{R}}_3 \cdot \tilde{\mathsf{R}}_2 \cdot \tilde{\mathsf{R}}_1 \\ &= \begin{pmatrix} x^2(1-\cos\varphi) + \cos\varphi & xy(1-\cos\varphi) - z\sin\varphi & xz(1-\cos\varphi) + y\sin\varphi & 0\\ xy(1-\cos\varphi) + z\sin\varphi & y^2(1-\cos\varphi) + \cos\varphi & yz(1-\cos\varphi) - x\sin\varphi & 0\\ xz(1-\cos\varphi) - y\sin\varphi & yz(1-\cos\varphi) + x\sin\varphi & z^2(1-\cos\varphi) + \cos\varphi & 0\\ 0 & 0 & 1 \end{pmatrix} \end{split}$$

- Standard description for a rotation used in computer graphics
- No gimbal lock
- Still not suitable for animations (meaningful interpolation between orientations not possible)

Some Recap of (Rotation with) Complex Numbers

A point p = (x, y)^T in the plane can be represented (using polar coordinates) as a complex number

$$\mathbf{p} = \|\mathbf{p}\|(\cos\alpha + i\sin\alpha) = \|\mathbf{p}\|e^{i\alpha}$$



• Multiplication of $\|\mathbf{p}\|e^{i\alpha}$ with $e^{i\beta}$ results in

$$\mathbf{q} = \|\mathbf{p}\|e^{i\alpha} \cdot e^{i\beta} = \|\mathbf{p}\|e^{i(\alpha+\beta)} = \|\mathbf{p}\|(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

With $x = \|\mathbf{p}\| \cos \alpha$, $y = \|\mathbf{p}\| \sin \alpha$ this can be written as

$$\begin{pmatrix} x'\\y' \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta\\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x\\y \end{pmatrix}$$



Quaternions

A quaternion



is the three-dimensional generalisation of a complex number

Discovered 1853 by W. R. Hamilton

Some important Properties of Quaternions

- Addition (associative, commutative, neutral element is $\mathbf{0} = [0, (0, 0, 0)]$) $\mathbf{q}_1 + \mathbf{q}_2 = [s_1 + s_2, \mathbf{v}_1 + \mathbf{v}_2]$
- Multiplication (not commutative, neutral element is $\mathbf{1} = [1, (0, 0, 0)]$) $\mathbf{q}_1 \cdot \mathbf{q}_2 = [s_1 \cdot s_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \ s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$
- Distributive $\mathbf{q}(\mathbf{r} + \mathbf{s}) = \mathbf{q}\mathbf{r} + \mathbf{q}\mathbf{s}$ and $(\mathbf{r} + \mathbf{s})\mathbf{q} = \mathbf{r}\mathbf{q} + \mathbf{s}\mathbf{q}$

• Length
$$\|\mathbf{q}\| = \sqrt{s^2 + x^2 + y^2 + z^2}$$

$$ar{\mathbf{q}} = [s, -\mathbf{v}]$$
 with $\mathbf{q} \cdot ar{\mathbf{q}} = s^2 + \|\mathbf{v}\|^2 = \|\mathbf{q}\|^2$

• Inverse

$$\mathbf{q}^{-1} = \|\mathbf{q}\|^{-2} \cdot \bar{\mathbf{q}} \quad (\|\mathbf{q}\| = 1 \implies \mathbf{q}^{-1} = \bar{\mathbf{q}})$$

Quaternions and Rotation

- A rotation around the axis ${\bf v}$ and the the angle φ can be described by a unit quaternion

$$\mathbf{q} = \left[\cos rac{arphi}{2}, \sin rac{arphi}{2} \cdot \mathbf{v}
ight], \; \|\mathbf{q}\| = 1$$

• Rotating a point **p** with a quaternion **q** is done by

$$R(\mathbf{p}) = \mathbf{q} \cdot \mathbf{p}_q \cdot ar{\mathbf{q}}$$
 with $\mathbf{p}_q = [0,\mathbf{p}]$

• Quaternions \mathbf{q} and $-\mathbf{q}$ (opposite direction and angle) describe the same roation, as

$$-\mathbf{q}\cdot\mathbf{p}_{q}\cdot\overline{-\mathbf{q}}=-1\cdot\mathbf{q}\cdot\mathbf{p}_{q}\cdot\overline{(-1\cdot\mathbf{q})}=-1\cdot\mathbf{q}\cdot\mathbf{p}_{q}\cdot-1\cdotar{\mathbf{q}}=\mathbf{q}\cdot\mathbf{p}_{q}\cdotar{\mathbf{q}}$$

• Concatenation of rotations

$$R_2(R_1(\mathbf{p})) = \mathbf{q}_2 \cdot (\mathbf{q}_1 \cdot \mathbf{p}_q \cdot ar{\mathbf{q}}_1) \cdot ar{\mathbf{q}}_2 = (\mathbf{q}_2 \cdot \mathbf{q}_1) \cdot \mathbf{p}_q \cdot \overline{(\mathbf{q}_1 \cdot \mathbf{q}_2)}$$

SLERP (Spherical Linear intERPolation)

- Linear interpolation (LERP)
 - → Given starting and end point \mathbf{p}_0 , \mathbf{p}_1 , interpolation parameter $t \in [0,1]$ LERP $(\mathbf{p}_0, \mathbf{p}_1, t) = (1 - t)\mathbf{p}_0 + t\mathbf{p}_1$
- Spherical linear interpolation (SLERP) \mathbf{p}_{0} LERP ($\mathbf{p}_{0}, \mathbf{p}_{1}, 0.2$) \mathbf{p}_{1}
 - LERP on the surface of a (unit) sphere

SLERP(
$$\mathbf{p}_0, \mathbf{p}_1, t$$
) = $\frac{\sin \left[(1-t)\Omega\right]}{\sin \Omega} \mathbf{p}_0 + \frac{\sin \left[t\Omega\right]}{\sin \Omega} \mathbf{p}_1$ (Using quaternions $\mathbf{q} = \left[\cos \frac{\varphi}{2}, \sin \frac{\varphi}{2} \mathbf{v}\right]$ this can be written as

SLERP $(\mathbf{q}_0, \mathbf{q}_1, t) = \mathbf{q}_0 \left(\mathbf{q}_0^{-1} \mathbf{q}_1 \right)^t = \mathbf{q}_0 \exp \left(t \cdot \log \left(\mathbf{q}_0^{-1} \mathbf{q}_1 \right) \right)$ with

$$\log\left(\mathbf{q}\right) = \left[0, \frac{\varphi}{2}\mathbf{v}\right], \ \exp\left(\left[0, \frac{\varphi}{2}\mathbf{v}\right]\right) = \left[\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2} \cdot \mathbf{v}\right] = \mathbf{q}$$

Similarity Transformations in \mathbb{P}^3

- Euclidean transformations and isotropic scaling (7 DOF)
 - Homogenous matrix vector representation

or $\mathbf{\tilde{x}}'$

$$\begin{pmatrix} x'\\y'\\z'\\1 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} s & 0 & 0\\0 & s & 0\\0 & 0 & s \end{pmatrix} \cdot \begin{pmatrix} r_{11} & r_{12} & r_{13}\\r_{21} & r_{22} & r_{23}\\r_{31} & r_{32} & r_{33} \end{pmatrix} \mathbf{T} \\ \mathbf{J} \end{bmatrix} \begin{pmatrix} x\\y\\z\\1 \end{pmatrix}$$
$$= \mathbf{H}_{S} \mathbf{\tilde{x}} = \begin{bmatrix} s\mathbf{R} & \mathbf{T}\\\mathbf{0}^{\top} & 1 \end{bmatrix} \mathbf{\tilde{x}} \text{ with } \mathbf{R}^{\top}\mathbf{R} = \mathbf{I} \text{ (and det}(\mathbf{R}) = 1)$$

Special Affine Transformations in \mathbb{P}^3

- Similarity transformations and anisotropic scaling (9 DOF)
 - Homogenous matrix vector representation

$$egin{pmatrix} x' \ y' \ z' \ 1 \end{pmatrix} = egin{bmatrix} s_x & 0 & 0 \ 0 & s_y & 0 \ 0 & 0 & s_z \end{pmatrix} \cdot egin{pmatrix} r_{11} & r_{12} & r_{13} \ r_{21} & r_{22} & r_{23} \ r_{31} & r_{32} & r_{33} \end{pmatrix} & \mathbf{T} \ y \ z \ 1 \end{pmatrix} egin{pmatrix} x \ y \ z \ 1 \end{pmatrix}$$

Special Affine Transformations in \mathbb{P}^3

Affine Transformations in \mathbb{P}^3

- General affine transformation (12 DOF)
 - Homogeneous matrix vector representation

$$egin{pmatrix} x' \ y' \ z' \ 1 \end{pmatrix} = egin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \ a_{21} & a_{22} & a_{23} & a_{24} \ a_{31} & a_{32} & a_{33} & a_{34} \ 0 & 0 & 0 & 1 \end{pmatrix} egin{pmatrix} x \ y \ z \ 1 \end{pmatrix}$$

can be written in the form (see also camera models)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \cdot \begin{bmatrix} \mathbf{R} & \mathbf{T} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

or
$$\mathbf{\tilde{x}}' = \mathbf{H}_A \mathbf{\tilde{x}} = \begin{bmatrix} \mathbf{A} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{\tilde{x}}$$

Projective Transformations in \mathbb{P}^3

- General projective transformation (15 DOF)
 - Homogeneous matrix vector representation

$$egin{pmatrix} x' \ y' \ z' \ w' \end{pmatrix} = egin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \ p_{21} & p_{22} & p_{23} & p_{24} \ p_{31} & p_{32} & p_{33} & p_{34} \ p_{41} & p_{42} & p_{43} & 1 \end{pmatrix} egin{pmatrix} x \ y \ z \ w \end{pmatrix}$$

can be written in the form

$$\begin{pmatrix} x'\\y'\\z'\\w' \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13}\\0 & b_{22} & b_{23}\\0 & 0 & b_{33} \end{pmatrix} \cdot \begin{bmatrix} \mathbf{R} & \mathbf{T} \end{bmatrix} \begin{pmatrix} x\\y\\z\\w \end{pmatrix}$$
or $\mathbf{\tilde{x}}' = \mathbf{H}\mathbf{\tilde{x}} = \begin{bmatrix} \mathbf{A} & \mathbf{T}\\p_{41} & p_{42} & p_{43} & 1 \end{bmatrix} \mathbf{\tilde{x}}$

Transformation	Matrix	# DOF	Preserves
translation	$[\mathbf{I} \mathbf{t}]_{3 imes 4}$	3	orientation
rigid (Euclidean)	$[\mathbf{R} \mathbf{t}]_{3 imes 4}$	6	length
similarity	$[s{f R} {f t}]_{3 imes 4}$	7	angles
affine	$[\mathbf{A} \mathbf{t}]_{3 imes 4}$	12	parallelism
projective	$[\mathbf{H}]_{4 imes 4}$	15	straight lines

- 3D transformations are defined analogously to 2D transformations
- 3 imes 4 matrices are extended with a fourth $[\mathbf{0}^ op \mathbf{1}]$ row for homogeneous transforms

Transformations on Co-Vectors

e = 4 ~ ~ SD

If a point $\tilde{\mathbf{x}}$ is transformed by a perspective 2D (3D) transformation $\tilde{\mathbf{H}}$ as

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}} \qquad (\tilde{\mathbf{\mu}}^{\mathsf{T}}\tilde{\boldsymbol{\mu}}')^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} \qquad (\tilde{\mathbf{\mu}}^{\mathsf{T}}\tilde{\boldsymbol{\mu}}')^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} \qquad (\tilde{\boldsymbol{\mu}}^{\mathsf{T}}\tilde{\boldsymbol{\mu}}')^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} \qquad (\tilde{\boldsymbol{\mu}}^{\mathsf{T}}\tilde{\boldsymbol{\mu}}')^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} \qquad (\tilde{\boldsymbol{\mu}}^{\mathsf{T}}\tilde{\boldsymbol{\mu}}')^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf{T}} = \tilde{\boldsymbol{\mu}}^{\mathsf$$

Thus, the action of a projective transformation on a co-vector such as a 2D line or 3D plane can be represented by the transposed inverse of the matrix. $(4^{T})^{2}(H^{T})^{\overline{C}}(H^{T})$

The transformation on a 3D line $\mathbf{\tilde{L}}$ is given by $\mathbf{\tilde{H}}\mathbf{\tilde{L}}\mathbf{\tilde{H}}^{ op}$

Plane at infinity

• If points on $ilde{\mathbf{m}}_{\infty} = (0,0,0,1)^{ op}$ are mapped by an affine transformation $ilde{\mathbf{H}}$ it holds

$$ilde{\mathbf{m}}_{\infty}' = (ilde{\mathbf{H}}^{-1})^{ op} ilde{\mathbf{m}}_{\infty} = egin{bmatrix} \mathbf{A}^{- op} & \mathbf{0} \ -\mathbf{T}^{ op} \mathbf{A}^{- op} & \mathbf{0} \ -\mathbf{T}^{ op} \mathbf{A}^{- op} & \mathbf{1} \end{bmatrix} egin{pmatrix} \mathbf{0} \ \mathbf{0} \ \mathbf{0} \ \mathbf{1} \end{pmatrix} = ilde{\mathbf{m}}_{\infty}$$

- The plane at infinity $\tilde{\mathbf{m}}_{\infty} = (0, 0, 0, 1)^{\top}$ is fixed under \mathbf{H} , iff \mathbf{H} is an affine transform
- Properties of the plane at infinity
 - $\,\circ\,\,$ Canonical position $\mathbf{\tilde{m}}_{\infty}=(0,0,0,1)^{ op}$
 - Contains the directions (vanishing points)
 - $\circ~$ Two planes are parallel iff their intersection line $ilde{f l}_\infty=(x_1,x_2,x_3,0)^ op$ is in $ilde{f m}_\infty$

 $\circ~$ A straight line is parallel to a straight line (or a plane) iff its intersection is in \tilde{m}_∞